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Nash Equilibria in Games and Simplicial Complexes

submitted by
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for the degree of
Doctor of Philosophy

University of Bath
Department of Computer Science

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Abstract

Nash's Theorem is a famous and widely used result in non-cooperative game theory which can be applied to games where each player's mixed strategy payoff function is defined as an expectation. Current proofs of this Theorem neither justify why this constraint is necessary or satisfactorily identifies its origins. In this Thesis we change this and prove Nash's Theorem for abstract games where, in particular, the payoff functions can be replaced by total orders. The result of this is a combinatoric proof of Nash's Theorem. We also construct a generalised simplicial complex model and demonstrate a more general form of Nash's Theorem holds in this setting. This leads to the realisation Nash's Theorem is not a consequence of a fixed-point theorem but rather a combinatoric phenomenon existing in a much more general mathematical model.

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I would like to take this opportunity to thank my supervisor, Professor Nicolai Vorobjov, for bringing out the best in me and helping me find an area of research which I have found to be interesting, enjoyable and compelling. I also want to acknowledge the advice and encouragement provided to me by other members of the department and fellow PhD students, especially Ros, Laura and Dalia. This just leaves those who have provided the emotional support and kept me going: Mum, Dad, Pete, Giles, Anna and Mina - thank you.

"Its significance was not immediately recognized, not even by the brash twenty-one-year-old author himself, and certainly not by the genius who inspired Nash, von Neumann" [Nasar, 1999]

Contents

List of Figures

Notation

Game Theory

$\Gamma = \Gamma_N$	N-player non-cooperative game
Γ_k	k-player non-cooperative game
S_i	Pure strategy set for player i
l_i	The cardinality of set S_i
s_i^l	Pure strategy belonging to S_i
S	Product space of all pure strategy sets i.e., $S_1 \times \cdots \times S_N$
$\mathbf{s}$	Pure strategy situation belonging to S
$H_i(\mathbf{s})$	Payoff awarded to player i for situation $\mathbf{s}$
$S^{(i)}$	$S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_N$
$\mathbf{s}^{(i)}$	Pure strategy for all player's bar player i i.e., a strategy from $S^{(i)}$
$(\mathbf{s}, s_i^{l'})$	Pure strategy situation $\mathbf{s}$ where player i uses pure strategy $s_i^{l'}$ instead of $s_i^l \in \mathbf{s}$, where $s_i^l, s_i^{l'} \in S_i$
$H_i(\mathbf{s}, s_i^{l'})$	Payoff awarded to player i for situation $(\mathbf{s}, s_i^{l'})$
P_i	Mixed Strategy set for player i

$p_i := (x_i^1, \dots, x_i^{l_i})$	Mixed strategy belonging to P_i Equivalently a probability distribution over the set S_i
x_i^l	Probability assigned to pure strategy s_i^l in p_i
P	Product space of all mixed strategy sets i.e., $P_1 \times \dots \times P_N$
$\mathbf{p} := (p_1, \dots, p_n)$	Mixed strategy situation from P
$G_i(\mathbf{p})$	Payoff awarded to player i for situation $\mathbf{p}$
$P^{(i)}$	$P_1 \times \dots \times P_{i-1} \times P_{i+1} \times \dots \times P_N$
$\mathbf{p}^{(i)}$	Mixed strategy for all player's bar player i i.e., an element from $P(i)$
$(\mathbf{p}, p_i^{l'})$	Mixed strategy situation $\mathbf{p}$ where player i uses mixed strategy $p_i^{l'}$ instead of $p_i^l \in \mathbf{p}$, where $p_i^l, p_i^l \in P_i$
$(\mathbf{p}, s_i^l)$	Mixed strategy situation $\mathbf{p}$ where player i uses pure strategy $s_i^l \in S_i$ instead of mixed strategy $p_i^l \in \mathbf{p}$, $p_i^l \in P_i$
$G_i^l := G_i(\mathbf{p}, s_i^l)$	Restricted payoff function i.e., Payoff awarded to player i for strategy $(\mathbf{p}, s_i^l)$
Z_i	Set of pure best responses for player i for a strategy $\mathbf{p^{(i)}} \in P^{(i)}$
Y_i	The support of mixed strategy $p_i \in P_i$
$P^{(N)}$	Subset of strategy space P defined by $P_1 \times \dots \times P_{N-1} \times \{s_N^{l_N}\}$ for $s_N^{l_N} \in S_N$.

Generalised Game Theory and Game Theory as a Simplicial Complex

$\Gamma^* = \Gamma_N^*$	Generalised N-player non-cooperative game
Γ_k^*	Generalised k-player non-cooperative game

$\succ_i$	Total order for the mixed strategy set P_i with regards to $P^{(i)}$
$\bar{B}_i(s_i^l)$	Covering element of $P^{(i)}$
$B_i(s_i^l)$	Connected covering element of P defined by $\bar{B}_i(s_i^l) \times P_i$
η_i	The nerve constructed from a covering over $P^{(i)}$
η	In an N-player game, the nerve of all N-coverings over P
η'	The nerve of the $(N-1)$-player subgame
Δ	General simplex or simplex from η (depending on context)
Λ	A face of Δ
W_i'	Set of vertices from η corresponding to covering elements with labels from S_i
$W_{\Delta,i}'$	The vertices from the simplex Δ which belong to the set W_i
$\Sigma_{\Delta,i}'$	The vertices from the simplex Δ corresponding to boundary faces with labels from S_i
$\mathbf{m'}$	Finite set of m' natural numbers
$\mathbf{m_i'} := \{m_{i-1}' + 1, \ldots m_i'\}$	Partition element of set $\mathbf{m'}$
ϕ'	Surjective map from V to $\mathbf{m'}$

Simplicial Complex Definition

$\mathbf{m}$	Finite set of m natural numbers
$\mathbf{m_i} := \{m_{i-1} + 1, \ldots, m_i\}$	Partition element of set $\mathbf{m}$
V	Set of vertices surjective to $\mathbf{m_1} \cup \cdots \cup \mathbf{m_n}$

V_k	Subset of V surjective to $\mathbf{m_1} \cup \cdots \cup \mathbf{m_k}$
ϕ	Surjective map from V to $\mathbf{m}$
ϕ_k	Surjective map from V_k to $\mathbf{m_1} \cup \cdots \cup \mathbf{m_k}$
W_i	Non-empty subset of vertices belonging to the interior of a simplicial complex with labels from $\mathbf{m_i}$
Σ_{m_i}	Simplex with vertex set $\mathbf{m_i}$
Σ, Δ	General simplex
σ	A face of Σ
$W_{\Delta,i}$	Vertices from Δ which belong to W_i
$\Sigma_{\Delta,i}$	Vertices from Δ belonging to Σ_{m_i}
$\mathcal{S} = \mathcal{S}_n$	Simplicial game complex of order n
$\mathcal{S}_k$	Simplicial game complex of order k
$\mathcal{S}^*$	Simplicial game complex defined by $\mathcal{S}_{21} * \mathcal{S}_{22}$
$\mathcal{S}_{1i}$	A simplicial complex defined by $\partial\Sigma_{m_1} * \cdots * \partial\Sigma_{m_{i-1}} * \partial\Sigma_{m_{i+1}} * \cdots * \partial\Sigma_{m_n}$
$\mathcal{S}_{2i}$	Arbitrary simplicial complex constructed from $\mathcal{S}_{1i}$ and W_i
$St(\sigma)$	Open star of σ
$\overline{St}(\sigma)$	Closure of $St(\sigma)$

General

$*$	Join operator

$\mathcal{C}^{(i)}$	Arbitrary pseudomanifold
d_i	Dimension of $\mathcal{C}^{(i)}$
$G = (V, E)$	Undirected graph G with vertices V and edges E
$\mathcal{N}$	Subset of vertices V (from G) defined to be normal
$v(\Sigma)$	Set of vertices of simplex Σ
$\partial\Sigma$	Boundary of simplex Σ

Chapter 1

Introduction

In 1950 when John Nash published his paper 'Equilibrium Points in N-Person Games' [Nash, 1950b] the long term consequences of his work were not immediately clear. However, since then game theory has become of increasing interest as its applications appear in fields as diverse as economics to evolutionary biology. This Thesis explores the now famous **Nash equilibria** and shows that, in fact, this phenomenon occurs in a much more general model of which non-cooperative game theory is a specific example.

1.1 Overview

A **game** is a mathematical model used to imitate real life situations where two or more people (players) make strategic decisions. Game theory then provides the tools and concepts needed to analyse such models and, in particular, provides reasoning to the strategic decisions each player should make. In light of this, the application of game theory is not restricted to traditional games, although games which do not rely on luck, for example *Chess* and *Naughts and Crosses*, are good examples of the theory. A wide variety of every-day events can be interpreted and analysed in terms of game theory, with examples including business mergers, predator-prey interactions, bidders at an auction eg 'Ebay', military strategies in times of conflict, evolutionary biology, economics ... the list is as diverse as it is long. Using [Hallinan, 2005],[Kelly, 2003], [Nasar, 1999], [Osborne, 2004] and [Thomas, 1984] and as our reference material we discuss the origins and basic concepts of this versatile tool.

1.2 History of Game Theory

Although game theory is a modern branch of mathematics, with the key developments being made in the 20th Century, the basic concepts have appeared in some shape or form for a much longer period of time. This section highlights some of the significant stages in the development of game theory which have ultimately shaped it into the diverse field it is today.

The term game theory was not officially coined until the 1900's. However the fundamental ideas can be traced back to the 17th Century and found in the solutions mathematicians put forward to help solve the gambling problems of the French nobility. As a result of the popular games of the time, for example *Chess*, development was confined to two player, **zero-sum games**: that is, games which either have a clear winner and looser, or they result in a tie. In 1713 formality was introduced to game theory by Pierre-Rémond de Montmort who defined the concept of a **minimax** solution for popular card game *Le Her*. This definition is still used today and describes the situation where each player chooses his strategy so to minimise his maximum loss. It was James Waldegrave, the games creator, who first identifiied the minimax solution of *Le Her*. This work was extended in 1738 by Daniel Bernoulli and included the introduction of the concept of a **payoff function**, used to determine the winner. Developments in game theory remained restricted to finding minimax solutions of two player games until the 1920's with the publication of a series of short papers by Émile Borel on gaming strategies.

It was John von Neumann who provided the major turning point in game theory. He introduced enough formality to allow game theory to be considered as its own independent mathematical discipline. Additionally von Neumann's work marks the first time game theory was used to solve long standing problems in economics. The first paper of significant importance published by von Neumann in this field was in 1928 [von Neumann, 1928][1] and was a proof of the so called **Minimax Theorem** for 2-player zero-sum games. This Theorem can be summarised as follows:

> In a 2-player, zero-sum game, each player can select an optimal strategy such that for a value v the maximum payoff awarded to player 1 given player 2's strategy is v and the maximum payoff awarded to player 2 given player 1's strategy is $-v$.

[1] Translated in "Contributions to the Theory of Games" [Tucker and Luce eds., 1959] as "On The Theory of Games of Strategy", pages 13-42

In particular each player maximises his own minimum payoff by minimising the maximum payoff which can be achieved by his opponent.

In 1937 von Neumann published his second significant contribution to game theory [von Neumann, 1937][2]. In this publication von Neumann had successfully found a solution to a long standing problem posed by Leon Walras in 1874 regarding the system of equations, influenced by supply and demand, governing production. The proof of this result is also a generalisation of the Minimax Theorem and was achieved by using Brouwer's Fixed Point Theorem. Von Neumann continued to develop his results and in doing so created the area of game theory. This culminated in 1944 with the collaboration with Oska Morgenstern and the publication of "Theory of Games and Economic Behavior" [von Neumann and Morgenstern, 1944]. A second edition appeared in 1947 and, despite this publication being the first text in this field, many of the ideas and concepts it contains are still used today.

John von Neumann was resident at Princeton for the majority of the time he devoted to game theory and shortly after the publication of "Theory of Games and Economic Behavior" it is was where von Neumann and John Nash, a student, crossed paths. While at Princeton, Nash attended a series of seminars on game theory and became captivated by the range of interesting and unsolved problems it contained. As an undergraduate student at the Carnegie Institute of Technology, Nash took an economics course which sparked original ideas of his own. However it was only after he became interested in game theory at Princeton did Nash develop these ideas further. In 1949 Nash approached Albert Tucker, who helped run the game theory seminars at Princeton, with the request of supervision for his Thesis. This request was duly accepted. Shortly after this Nash visited von Neumann to discuss his idea for an equilibrium point in a game with more than 2 players. Before Nash was allowed to fully explain his proof and reach his conclusion it is reported von Neumann interrupted abruptly and said

> "That's trivial, you know. That's just a fixed point theorem" [Nasar, 1999][3]

The two mathematicians had approached N-person game theory from two opposing angles. Von Neumann's work emphasised the importance of coalitions while Nash focused on independence among players. Fortunately Nash communicated his ideas of a generalisation of von Neumann's work to games with multiple players to David Gale who immediately recognised its significance and became an ambassador for Nash. Interestingly von Neu-

[2] Translated in "A Model of General Economic Equilibrium" [von Neumann, 1945]
[3] As told to Harold Kuhn

mann's rejection of Nash's idea combined with Tucker's request of amendments to his Thesis almost saw Nash abandon his theorem to tackle a problem in algebraic geometry. Thankfully, Tucker was able to persuade Nash to continue with his original Thesis and in 1950 Nash published his work on equilibrium situations [Nash, 1950b]. This was quickly replaced with a modification in 1951 [Nash, 1951] which, like the work by von Neumann, uses Brouwer's Fixed Point Theorem. Around this time Nash also produced a third paper in game theory, this time describing the Bargaining Problem [Nash, 1950a].

The foundations of non-cooperative game theory are regarded to rest upon the theorems by von Neumann in 1928 and Nash in 1950. Nash considered his result as a direct extension of von Neumann's work, and given von Neumann's reported reaction it is likely he shared Nash's opinion on this. However, Nash's work also provided a deviation. Von Neumann's work, while important, focused on the 2-player zero-sum case which has little relevance to the mathematical models of real life situations.

It is not a coincidence the development of game theory coincided with the Second World War. During this period some of the greatest mathematicians and scientists of the time were assembled to help build and develop sophisticated weaponry. When weapons had become too complicated to be used efficiently attention was turned to strategy. At this time the work on 2-player zero-sum games was the only theory which was complete and could be seen as reasonable to work from. Such games rely on the concept 'my win your gain' and some mathematicians started to doubt how well this model fitted the real life scenarios they were studying. With the development of more advance weaponry, including nuclear bombs, it became clear an outright war was not a realistic solution. News of Nash's work was therefore met with great excitement. Finally there was a model which was not required to identify a clear 'winner' and 'loser' and allowed independent decisions to be made without the need for cooperation with the 'enemy'. Since then interest has been wide and diverse with the game theory model being used in a range of surprising disciplines including politics, and psychology.

In 1994 John Nash shared the Nobel Prize for his work on Nash Equilibria with mathematician John C. Harsanyi and economist Reinhard Selten who said

> "Nobody would have foretold the great impact of the Nash equilibrium on economics and social science in general. It was even less expected that Nash's equilibrium point concept would ever have any significance for biology theory" [Nasar, 1999]

1.3 Terminology - Non-cooperative Game Theory

We now introduce game theory formally and discuss the definitions and concepts which will be important in this Thesis.

Consider a scenario involving N people where each person has to choose one of a finite number of options (or strategies) to pursue which may have the affect of influencing the outcome. Then in the game theory model each person is referred to as a player and a game is described once every player has selected a strategy to 'play'. When each player chooses his strategy with the sole purpose of optimising his own personal outcome alone then the game is said to be **non-cooperative**. In the N-player non-cooperative game define S_i to be the set of strategic decisions or **pure strategies** available to each player $i = \{1, \dots, N\}$ where $|S_i| = l_i$ is finite. If player i chooses strategy $s_i^j \in S_i$ then define

$$\mathbf{s} \;:=\; (s_1^{j_1}, \dots, s_N^{j_N}) \;\in\; S_1 \times \cdots \times S_N \tag{1.1}$$

where for all $i = \{1, \dots, N\}$ we have $j_i \in \{1, \dots, l_i\}$ and call $\mathbf{s}$ a **strategy profile** or **situation** of the game. For every situation $\mathbf{s} \in S \;:=\; S_1 \times \cdots \times S_N$ each player is awarded a personal outcome. This outcome can be assigned a numerical value which can be calculated for each player i in accordance to his personal utility or **payoff function**:

$$H_i(\mathbf{s}) : S_1 \times \cdots \times S_N \mapsto \mathbb{R} \tag{1.2}$$

For example, the payoff function maybe a measure of how much money a player has gained or lost. It is assumed each player's objective is to maximise their payoff function therefore, while the interpretations of the payoffs will be different for each game, success is generally recongised when the highest (or lowest) payoff value is achieved. If all players know every player's payoff function then the game is said to be one of **complete information**. For the purpose of this Thesis this property will be assumed in all games discussed.

Game theory models situations where each player's personal outcome will depend on how he interacts with all other players. Therefore each player will be hoping to select a strategy which will result in the outcome of the game (or situation being modelled) being in their favour. This selection process is inevitably non-trivial and becomes more complicated as the number of players and strategies available increases.

1.3.1 Optimal Strategies

For every player $i = \{1, \ldots, N\}$ the situation $\mathbf{s} \in S$ contains a pure strategy from S_i which player i hopes will maximise his payoff function $H_i(\mathbf{s})$. However this strategy choice may result in an opponent being unable to attain his maximum payoff. This is true for all players i and consequently the strategies selected by each player will not only affect their personal payoff but also that of their opponents.

It can be assumed all players are working with the aim of maximising their payoff. Since all players are trying to achieve this, it is highly probable each player's payoff will be significantly less than had been anticipated. In 1950 John Nash [Nash, 1950b] introduced the concept of an **equilibrium situation** (or **equilibrium point**) which addresses this problem. An equilibrium point is a situation of the game where no player can achieve an improved payoff by altering his strategy choice alone. Equivalently each player's payoff is maximal with regards to the strategies played by his opponents. Observe this may not result in each player achieving his most maximum payoff value, but since no player has any incentive to change their strategy the outcome is one which will result in all players being 'happy'. Let $(\mathbf{s}, s_i^{j'})$ be the situation where the pure strategy chosen by player i in $\mathbf{s}$ is replaced by $s_i^{j'} \in S_i$. Then the formal definition of Nash equilibria is as follows:

Definition 1.1 (Nash Equilibrium)
A situation $\mathbf{s}$ *satisfies the conditions of* Nash equilibrium *if for all players i and all* $s_i^{j'} \in S_i$

$$H_i(\mathbf{s}) \geq H_i(\mathbf{s}, s_i^{j'}) \tag{1.3}$$

For player i, and all $s_i^{j'} \in S_i$, this is equivalent to identifying a situation $\mathbf{s} \in S := S_1 \times \cdots \times S_N$ which satisfies the set of l_i inequalities given in (1.3). Then for all players, solving the resulting $\sum_{i=1}^{N} l_i$ inequalities will determine any equilibrium points which exist in the game. An example of Nash equilibrium is given in Section 1.3.3.

1.3.2 Bimatrix Games

In a 2 player game each player's payoff function can be expressed as a matrix. Consequently 2-player games are often referred to as **bimatrix games**. If player 1 has two pure strategies and player 2 has three such that $S_1 = \{s_1^1, s_1^2\}$ and $S_2 = \{s_2^1, s_2^2, s_2^3\}$ then the payoff matrix for player i, $i = \{1, 2\}$, is given by

$$A_i = \begin{pmatrix} H_i(s_1^1, s_2^1) & H_i(s_1^1, s_2^2) & H_i(s_1^1, s_2^3) \\ H_i(s_1^2, s_1^1) & H_i(s_1^2, s_2^2) & H_i(s_1^2, s_2^3) \end{pmatrix} \tag{1.4}$$

Where in this representation player 1's strategies are the rows of the matrix and player 2's are the columns. In reality it does not matter which way round these two are written, provided the same order is used for the corresponding matrix for the second player. In the case

$$A_1 = -A_2 \tag{1.5}$$

we have a **zero-sum** game which is equivalent to

$$A_1 + A_2 = 0 \tag{1.6}$$

and the reason for the name given to this subset of games is clear. Since in this case a single matrix can be used to define the whole game such games are referred to simply as **matrix games**. The game *Naughts and Crosses* is an example of a zero-sum game.

Observe the dimension of the matrix A_i is $l_1 \times l_2$ (where recall $|S_i| = l_i$). Therefore the extension of this to represent a 3-player game would result in 'matrices' of dimension $l_1 \times l_2 \times l_3$. Similarly for a game consisting of N-players. Consequently choosing to express the payoff values in this form for games with three or more players does not provide an easy or useful representation of this information. In a bimatrix game the two matrices A_1 and A_2 are often expressed in one table given in Figure 1.1.

$H_1(s_1^1, s_2^1),\ H_2(s_1^1, s_2^1)$	$H_1(s_1^1, s_2^2),\ H_2(s_1^1, s_2^2)$	$H_1(s_1^1, s_2^3),\ H_2(s_1^1, s_2^3)$
$H_1(s_1^2, s_2^1),\ H_2(s_1^2, s_2^1)$	$H_1(s_1^2, s_2^2),\ H_2(s_1^2, s_2^2)$	$H_1(s_1^2, s_2^3),\ H_2(s_1^2, s_2^3)$

Figure 1.1: Example payoff table

The first element in each position of the table in Figure 1.1 refers to the payoff awarded to player 1 and the second to player 2.

1.3.3 Prisoner's Dilemma

The *Prisoner's Dilemma* is a popular example of game theory and was devised to investigate the psychology behind ethics and in particular the conflict between morality and self-interest. The example involves the arrest of two suspects of a serious crime who make a pact not to confess their guilt. While there is sufficient evidence to convict each individually of other smaller crimes the police do not have enough proof to charge them both for the more serious crime unless one acts as an informant and confesses. In light of this the police offer both prisoners the same deal. If one prisoner testifies and one remains quiet, the informant will escape a jail term while the other prisoner will receive the maximum 10 years. If neither prisoner testifies then they will both receive a jail term of 1 year and in the case that both prisoners testify against each other then they both will be sentenced to 5 years. Neither prisoner will have information regarding the choice the other is making, so the dilemma is do the prisoners testify or keep quiet?

Remark
Games, like this, where each player has the option of just two (pure) strategies are called **dyadic**.

The table in Figure 1.2 summarises the 'payoffs', or jail terms handed to each prisoner, for each scenario using the representation given in Figure 1.1.

	Prisoner 2 - quiet	Prisoner 2 - testify
Prisoner 1 - quiet	$(1, 1)$	$(10, 0)$
Prisoner 1 - testify	$(0, 10)$	$(5, 5)$

Figure 1.2: Payoff table for the game *Prisoner's dilemma*

In Figure 1.2 if the payoff is given by (a, b) then prisoner 1 will be jailed for a years and prisoner 2 for b years. If both prisoners are working together and aiming to keep both jail terms to a minimum then both prisoners realise they should remain quiet. However problems arise when either one of the prisoners believe the other may break the original pact in the hope they will escape jail and secure a better position for themselves. In this case we have a non-cooperative game as each player is only focused on improving his outcome. Given neither prisoner will know the choice the other makes, what decision should each prisoner make?

Neither prisoner can predict what the other will do - if they can then the decision is easy. If prisoner 1 knows prisoner 2 will stay silent then prisoner 1's best option is to testify and avoid a prison sentence. Surprisingly if prisoner 1 knows prisoner 2 will testify then again his best option is to testify and reduce the jail term he will receive. Therefore betraying is the best option in both of these cases and is the **dominant strategy**. Prisoner 2 thinks similarly so both choose to testify. This is an equilibrium point of the game as neither player can reduce his jail term by altering his decision to testify alone. Interestingly this is the only equilibrium point of the game, since when both prisoners remain quiet either one can improve their jail term by testifying, the same is true (for one player) when one testifies and one does not. In this example breaking the pact to improve the individual resulted in both prisoners being worse off. In particular this very nicely demonstrates how a Nash equilibrium situation may not be the perfect result. What is interesting about this result is that both prisoners will choose to testify realising the other prisoner will have reached the same decision using the same analysis. Despite this they both still testify knowing that both keeping quiet would lead to a better outcome.

1.3.4 Mixed Strategies

By definition Nash equilibrium points are those situations where, for the given situation, each player is content in the payoff value they have been awarded. Consequently such situations are often regarded as the solution set to a game. However within our current definition of a game there is no guarantee such equilibrium situations exist.

Example 1.2 (Rock-Paper-Scissors)
The game *Rock-Paper-Scissors* is traditionally a two player game where each player's strategy set is given by $\{r$=rock, p=paper, s=scissors$\}$. The rules of the game are rock beats scissors, scissors beats paper and paper beats rock. If both players chose the same strategy the game is a draw. Let 1 denote a win, 0 a draw and -1 a loss then the payoff

table for this game is as given in Figure 1.3

P1 / P2	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Figure 1.3: Payoff table for the game *Rock-Paper-Scissors.*

Observe this game contains no equilibrium situation.

(End Example)

The problem with the game *Rock-Paper-Scissors* is that both players are required to out guess their opponent. Another example of this would be a simplified version of poker where each player is required to 'bluff'. This guessing introduces uncertainty into a game and in such cases there will be no Nash equilibrium situation. To interpret this scenario the notion of a **mixed strategy** is introduced. To define this formally we require the definition of a **probability distribution** and **discrete random variable**:

Definition 1.3 (Probability Distribution)
A probability distribution *is a vector of probabilities* $X = (x_1, \ldots, x_n)$ *where for every* $i = \{1, \ldots, n\}$, $x_i \geq 0$ *and*

$$\sum_{i=1}^{n} x_i = 1 \tag{1.7}$$

Definition 1.4 (Discrete Random Variable)
A discrete random variable *is a pair of vectors, one containing a set of outcomes* $A = (a_1, \ldots, a_n)$ *and a probability distribution* $X = (x_1, \ldots, x_n)$ *where* x_i *is the probability of outcome* a_i *occurring.*

Then a **mixed strategy** is defined as:

Definition 1.5 (Mixed Strategy)
A mixed strategy *for player* i *is a discrete random variable or equivalently a probability distribution over his set of pure strategies* S_i.

Returning to the original motivation for a mixed strategy observe the probability distribution reflects the uncertainty player's exhibit towards the strategy choice made by an opponent. For each player i let a mixed strategy situation p_i be defined by

$$p_i \;:=\; (x_i^1, \dots, x_i^{l_i}) \tag{1.8}$$

where as before $l_i = |S_i|$ and each x_i^j is the probability pure strategy $s_i^j \in S_i$ is chosen. Denote the set of all mixed strategies (probability distributions) for player i by P_i where the each mixed strategy can be represented as a vector in Euclidean space. Let the product space of all mixed strategies be P where

$$P \;:=\; P_1 \times \cdots \times P_N \tag{1.9}$$

and then a **(mixed strategy) situation** of the game is given by

$$\mathbf{p} \;:=\; (p_1, \dots, p_N) \in P \tag{1.10}$$

Definition 1.6 (Support)
The support $Y_i \subset S_i$ *of a mixed strategy* $p_i \in P_i$ *is the non-empty subset of pure strategies assigned a non-zero probability in* p_i*.*

Definition 1.7 (Totally Mixed Strategy)
For mixed strategy $\mathbf{p}$ *and all players* $i = \{1, \dots, N\}$*, if the support* Y_i *for player* i *satisfies* $Y_i = S_i$ *then* $\mathbf{p}$ *is a* totally mixed strategy*.*

Example 1.8
Returning to Example 1.2. For player 1 let $p_1 = (x_1^1, x_1^2, x_1^3)$ then for any game x_1^1, x_1^2 and x_1^3 are the probabilities player 1 chooses rock, paper or scissors respectively.

(End Example)

For a fixed $j \in \{1, \dots, l_i\}$ consider the mixed strategy p_i with $x_i^j = 1$ and $x_i^{j'} = 0 \;\; \forall j' \in \{1, \dots, l_i\}$ with $j \neq j'$. This is equivalent to the vector $(0, \dots, 1, \dots, 0)$ where the value 1 lies in position j and is identified with player i selecting pure strategy $s_i^j \in S_i$. This is true for all $j \in \{1, \ldots, l_i\}$ and thus we must have $S_i \subset P_i$. In particular the situations where players do have enough information and confidence to predict the strategy choice made by an opponent with 100% certainty is still represented within this definition. Since p_i is a probability distribution the following relation is true

$$x_i^1 + \cdots + x_i^{l_i} \;=\; 1 \tag{1.11}$$

and without loss of generality rearranging gives

$$x_i^{l_i} \;=\; 1 - (x_i^1 + \cdots + x_i^{(l_i - 1)}) \tag{1.12}$$

Consequently if each of the pure strategies belonging to S_i are considered as variables then equation (1.12) tells us the probability assigned to one variable is dependent on all others. Further because each x_i^j represents a probability it must be that $0 \leq x_i^j \leq 1$ for all $j = \{1, \ldots, l_i\}$. Using this with equation (1.11) the set P_i has a geometric interpretation. From (1.12) we know only $(l_i - 1)$ of the variables from S_i take independent values therefore we construct P_i over a space of dimension $(l_i - 1)$.

Given $S_i \subset P_i$ the set of pure strategies must be identifiable within the geometric representation of P_i. Such strategies are described by the set

$$\{x_i^j = 1,\, x_i^k = 0 \mid \forall\;\, j,\, k \in \{1, \ldots, l_i\},\; k \neq j\} \tag{1.13}$$

which identifies l_i unique points within the geometric object. The geometric representation of P contains exactly all mixed strategies and so in particular must contain all 1-dimensional edges given by the set

$$\{x_i^j + x_i^{j'} = 1,\, x_i^k = 0 \mid \forall\;\, j, j', k \in \{1, \ldots, l_i\},\; j \neq j' \neq k\} \tag{1.14}$$

In this set note when $x_i^{j'} = 0$ we have the set given in (1.13) and in particular the pure strategy situations are the end points of the 1-dimensional edges given in (1.14). Then observe the resulting geometric representation of the mixed strategy set P_i is equivalent to the convex hull of all pure strategies in S_i (as represented in (1.13)), or equivalently a simplex of dimension $(l_i - 1)$. Figure 1.4 illustrates this with an example of the simplex P_i when $l_i = 2$ and $l_i = 3$.

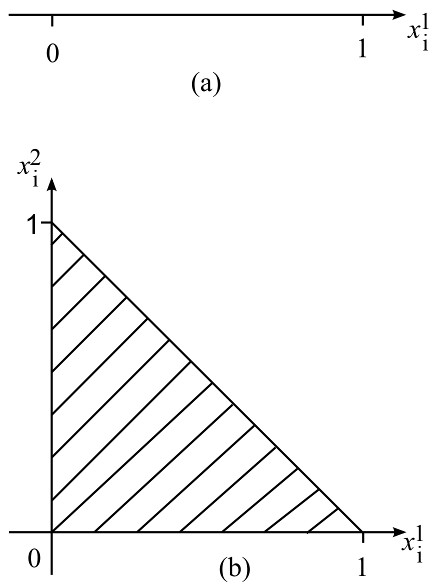


Figure 1.4: Constructing P_i as a geometric object where in (a) $l_i = 2$ and (b) $l_i = 3$

Remark
Observe for all $i = \{1, \dots, N\}$ the mixed strategy set P_i is infinite.

For a mixed strategy situation denote the payoff assigned to player i by

$$G_i : P_1 \times \cdots \times P_N \mapsto \mathbb{R} \tag{1.15}$$

Then the payoff value G_i is an **expectation** of the payoffs H_i.

Definition 1.9 (Expectation)
For a discrete random variable of events $A = (a_1, \dots, a_n)$ with probability distribution $X = (x_1, \dots, x_n)$ the expected outcome*, or* expectation *is given by*

$$\sum_{i=1}^{n} x_i \cdot a_i \tag{1.16}$$

Observe the payoff functions H_i take as input a value from S. Therefore for a 2 player game with $S_i = \{s_i^1, \ldots, s_i^{l_i}\}$ and mixed strategy $p_i = (x_i^1, \ldots, x_i^{l_i})$ for $i \in \{1, 2\}$ the expected payoff awarded to player 1 is given by

$$G_1 \;:=\; \sum_{j=1}^{l_1} \sum_{k=1}^{l_2} x_1^j \,\cdot\, x_2^k \,\cdot\, H_1(s_1^j, s_2^k) \tag{1.17}$$

This extends to all finite number of players N. In particular G_i in an N-player game is defined as

$$G_i \;:=\; \sum_{i_1=1}^{l_1} \cdots \sum_{i_N=1}^{l_N} x_1^{i_1} \,\cdot\, \cdots \,\cdot\, x_N^{i_N} \,\cdot\, H_i(s_1^{i_1}, \ldots, s_N^{i_N}) \tag{1.18}$$

Remark
When $\mathbf{p} = \mathbf{s}$ then $G_i(\mathbf{p}) = H_i(\mathbf{s})$.

Notation
Let Γ *denote the* N*-player non-cooperative game with mixed strategy sets* P_i *and payoff functions* G_i*.*

Let $(\mathbf{p}, p_i')$ denote the situation where the mixed strategy chosen by player i in $\mathbf{p}$ is replaced by $p_i' \in P_i$. Then a Nash equilibrium point in game Γ is defined to be:

Definition 1.10 (Nash Equilibrium for Mixed Strategies)
For a non-cooperative game Γ*, a situation* $\mathbf{p}$ *is said to be a* Nash equilibrium situation *if for all player's i*

$$G_i(\mathbf{p}) \geq G_i(\mathbf{p}, p_i') \qquad \forall\, p_i' \in P_i \tag{1.19}$$

Once again Nash equilibrium points will be referred to as **equilibrium points** or **equilibrium situations**.

Example 1.11
Continuing from Example 1.8. When player 1's mixed strategy is given by $x_1^1 = x_1^2 = x_1^3 = \frac{1}{3}$ and if player 2 selects the same mixed strategy the resulting situation is a Nash equilibrium situation.

(End Example)

Every non-cooperative game Γ can be further sub-categorised depending on the characteristics of its payoff functions. In particular any game is said to be either **non-degenerate** or **degenerate**. We return to this in Section 1.4.

It is not obvious that within any game Γ we should be able to identify an equilibrium point. However this is exactly what Nash's Theorem allows us to assume.

Theorem 1.12 (Nash's Theorem [Nash, 1950b])
Every non-cooperative game Γ has at least one equilibrium situation.

This result was proved differently for bimatrix games by Lemke and Howson in 1964 [Lemke and Howson Jr, 1964] and in 1971 Rosenmüller [Rosenmüller, 1971] and Wilson [Wilson, 1971] both independently achieved the same result for general non-cooperative N-person games. Collectively their work has resulted in the following, more common, version of Nash's Theorem

Theorem 1.13 (Nash's Theorem - Extension)
In every non-cooperative game Γ there exists at least one equilibrium situation. When Γ is non-degenerate then the number of equilibrium situations is finite and odd.

An overview of current proofs of Theorem 1.12 and 1.13 are given in Section 1.7. From this point forward Nash's Theorem will refer to the formulation given in Theorem 1.13.

1.3.5 Best Response Correspondences

First some important notation

Notation

$$\mathbf{s^{(i)}} \in S^{(i)} := S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_N \tag{1.20}$$

$$\mathbf{p^{(i)}} \in P^{(i)} := P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_N \tag{1.21}$$

Remark
The notation $P^{(i)}$ is equivalent to the notation P_{-i} popular in game theory publications.

An important tool used to determine equilibrium situations in a non-cooperative game Γ are the set of **best response correspondences**. First notice we can define a situation $\mathbf{p} \in P$ by $\mathbf{p} := \mathbf{p^{(i)}} \times \{p_i\}$, where $\mathbf{p^{(i)}} \in P^{(i)}$ and $p_i \in P_i$.

Definition 1.14 (Best Response Correspondence)
For all $i = \{1, \ldots, N\}$ and for every situation $\mathbf{p^{(i)}} \in P^{(i)}$ player i's best response correspondence A_i *is defined by*

$$A_i(\mathbf{p^{(i)}}) := \{\mathbf{p^{(i)}} \times \{p_i\} \mid G_i(\mathbf{p^{(i)}} \times \{p_i\}) \geq G_i(\mathbf{p^{(i)}} \times \{p_i'\}) \;\forall\; p_i' \in P_i\} \tag{1.22}$$

for $p_i \in P_i$

Remark

Then situation $\bar{\mathbf{p}} = (\bar{p}_1, \ldots, \bar{p}_i, \ldots, \bar{p}_N)$ is a Nash equilibrium point of Γ when $\bar{\mathbf{p}}^{(\mathbf{i})} \times \{\bar{p}_i\}$ belongs to $A_i(\bar{\mathbf{p}}^{(\mathbf{i})})$, and in particular $\bar{\mathbf{p}} \in A_i$, for every player i. Equilibrium situations can therefore be identified as the elements belonging to

$$A_1 \cap \cdots \cap A_N \tag{1.23}$$

Equivalently equilibrium situations occur at the points of intersection of all functions A_i. This clearly becomes increasingly more complicated as the number of players, and their strategy sets, increase in size.

The use of best response correspondences to identify the set of equilibrium situations leads to an important observation. Suppose for a fixed situation $\mathbf{p^{(i)}} \in P^{(i)}$ player i's optimal strategy is mixed strategy p_i with support Y_i. Then every pure strategy in Y_i must also be an optimal strategy for player i for situation $\mathbf{p^{(i)}}$. For contradiction assume this is not the case. Let $Y_i = \{s_i^1, s_i^2, s_i^3\}$ and assume the payoff awarded to player i when he chooses these pure strategies are a, a, b respectively, where $a > b$. Then if $x_i^1 = x_i^2 = x_i^3 = \frac{1}{3}$ the expected payoff earned by player i is $\frac{a+a+b}{3} < a$. If however player i changes his strategy to $x_i^1 = x_i^2 = \frac{1}{2}$ and $x_i^3 = 0$ then player i's expected payoff is now $\frac{a+a}{2} = a$. Therefore the mixed strategy originally chosen does not correspond to an optimal strategy and this contradicts our initial assumption. This holds for all probability distributions of x_i^1, x_i^2, x_i^3 where $x_i^3 \neq 0$.

This leads to an important characterisation to the definition of a mixed strategy equilibrium point, Definition 1.10. In this definition an equilibrium point $\mathbf{p} = (p_1, \dots, p_i, \dots, p_N)$ is required to satisfy the following set of infinite inequalities:

$$G_i(\mathbf{p}) \geq G_i(\mathbf{p}, p_i') \qquad \forall \;\; p_i' \in P_i \tag{1.24}$$

For all $i = \{1, \dots, N\}$, since the probability distribution $p_i \in \mathbf{p}$ is an optimal strategy for player i, the support of p_i must only contain those pure strategies from S_i which are optimal pure strategies for player i. In light of this, the condition for $\mathbf{p}$ to be an equilibrium situation is equivalent to $\mathbf{p}$ satisfying the following set of *finite* inequalities:

$$G_i(\mathbf{p}) \geq G_i(\mathbf{p}, s_i^j) \qquad \forall \;\; s_i^j \in S_i \tag{1.25}$$

In particular a mixed strategy belongs to the best response correspondence if and only if all pure strategies in its support are pure best responses to the same situation. Geometrically player i's best responses to a situation from $P^{(i)}$ must be a face of P_i.

Identifying Optimal Strategies

For an N-player non-cooperative game Γ the first step in constructing the best response correspondence for each player $i = \{1, \ldots, N\}$ is to identify player i's optimal strategies in response to all situations from $P^{(i)}$.

For every payoff function G_i and for all pure strategies $s_i^j \in S_i$ consider the set of l_i **restricted payoff functions** given by

$$G_i^j : P_1 \times \cdots \times P_{i-1} \times \{s_i^j\} \times P_{i+1} \times \cdots \times P_N \mapsto \mathbb{R} \tag{1.26}$$

for all $i \in \{1, \ldots, N\}$ and $s_i^j \in S_i$ where

$$G_i^j(\mathbf{p^{(i)}}) \;=\; G_i(\mathbf{p}, s_i^j) \tag{1.27}$$

As the strategies from $P^{(i)}$ vary the function $G_i^j(\mathbf{p^{(i)}})$ produces a different real valued output. Consequently this function can be plotted in cartesian co-ordinates with an axis for each of the independent strategies (variables) from $S^{(i)}$ and one for the payoff awarded to player i. Repeating for all $s_i^j \in S_i$ produces l_i distinct graphs which can be compared to identify player i's optimal strategies for all situations from $P^{(i)}$. This information is then represented over P to produce player i's best response correspondence. The following example illustrates this process and the use of best response correspondences to identify equilibrium situations.

Example 1.15
For a 2-player dyadic game Γ let the mixed strategy for player i in $\{1, 2\}$, be denoted by $p_i = (x_i^1, x_i^2)$. By equation (1.12) we have $x_i^2 = 1 - x_i^1$ and $p_i = (x_i^1,\, 1 - x_i^1)$. Consequently each player has just one independent pure strategy and the simplices P_1 and P_2 are of dimension 1. Since each player has just one independent strategy denote the mixed strategy strategies for player 1 and player 2 by x_1 and x_2 respectively.

By definition the payoff functions G_i, $i \in \{1, 2\}$, are expectations of the payoff functions H_i. Assume the payoff function H_1 is given by the matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{1.28}$$

with the corresponding matrix for H_2 being

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \tag{1.29}$$

Then the payoff functions G_1 and G_2 are given by

$$G_1 : P_1 \times P_2 \mapsto a_1x_1x_2 + a_2x_1(1-x_2) + a_3(1-x_1)x_2 + a_4(1-x_1)(1-x_2) \tag{1.30}$$

$$G_2 : P_1 \times P_2 \mapsto b_1x_1x_2 + b_2x_1(1-x_2) + b_3(1-x_1)x_2 + b_4(1-x_1)(1-x_2) \tag{1.31}$$

Remark
The payoff functions given in equations (1.30) and (1.31) are **polylinear**. That is each term is linear with respect to each player's strategy set S_i.

Focus initially on determining the optimal payoffs for player 1 and so in particular consider the restricted payoff functions G_1^j for pure strategies $s_i^j \in S_i,\ j \in \{1,2\}$. For mixed strategy situation $\mathbf{p}$ the restricted payoffs for player 1 are

$$G_1^1 = G_1(\mathbf{p}, s_1^1) : \{s_1^1\} \times P_2 \mapsto a_1x_2 + a_2(1-x_2) = x_2(a_1-a_2) + a_2 \tag{1.32}$$

$$G_1^2 = G_1(\mathbf{p}, s_1^2) : \{s_1^2\} \times P_2 \mapsto a_3x_2 + a_4(1-x_2) = x_2(a_3-a_4) + a_4 \tag{1.33}$$

Observe equations (1.32) and (1.33) are linear. Assume $a_4 < a_1 < a_2 < a_3$ then Figure 1.5 shows the two restricted payoff functions.

From Figure 1.5 when $x_2 < y_0$ the payoff function G_1^1 produces player 1's optimal payoff; equivalently player 1's optimal strategy is pure strategy s_1^1. When $x_2 > y_0$ payoff function

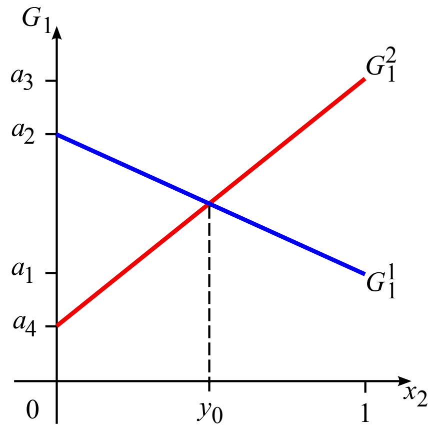


Figure 1.5: Player 1's two restricted payoff funtions

G_1^2 is optimal and player 1's optimal strategy is pure strategy s_1^2. Finally when $x_2 = y_0$ both payoff functions are optimal and player 1's optimal strategy will be a totally mixed strategy. This information is translated to form the best response correspondence for player 1. This is shown in Figure 1.6

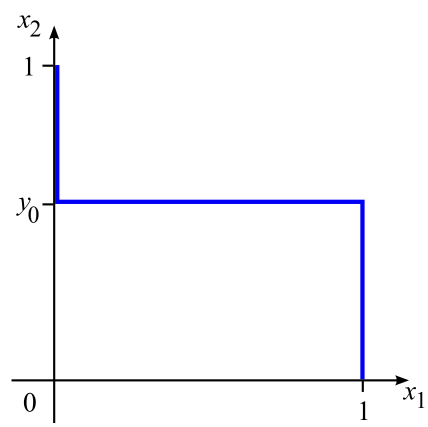


Figure 1.6: Player 1's best response correspondence

The same procedure can be repeated for player 2 and once again the two restricted payoff functions G_2^j, for $j \in \{1, 2\}$ are 1-dimensional. Without loss of generality let the relation $b_4 < b_1 < b_2 < b_3$ be true then the plot of the two restricted payoff functions is the same as that given in Figure 1.5 with $a_j = b_j$ for all $j \in \{1, 2, 3, 4\}$ and the point y_0 labelled by z_0. As before construct player 2's best response correspondence. This is given in Figure 1.7

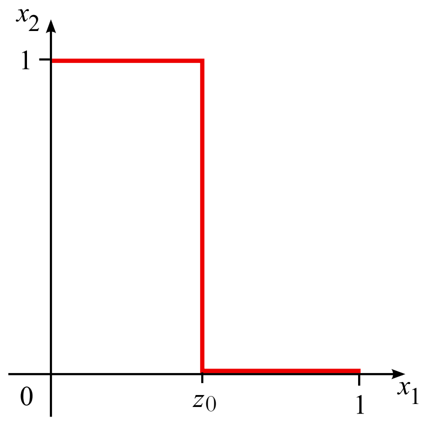


Figure 1.7: Player 2's best response correspondence

Each point of intersection of the best response correspondences for player's 1 and 2 correspond to a situation from Γ where neither player will benefit by playing an alternative strategy and as such must correspond to an equilibrium situation of the game. Figure 1.8 shows the intersection of the best response correspondences for player 1 and 2. There are clearly three points of intersection and thus there are three equilibrium situations; $(1, 0)$ and $(0, 1)$, represent pure strategy equilibrium situations while the point (z_0, y_0) corresponds to a totally mixed strategy situation.

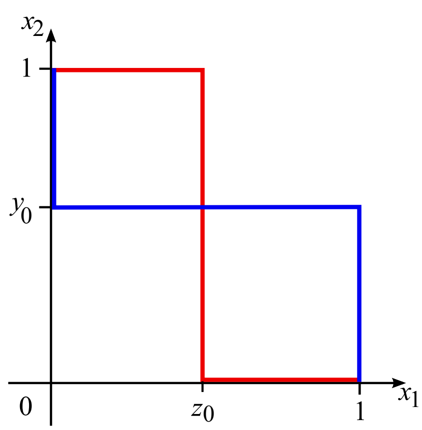


Figure 1.8: The intersection of the best response correspondences for player's 1 and 2

In a non-degenerate 2-player dyadic game each player has just two possible distinct best response correspondences. Figure 1.9 illustrates the three remaining interesting configurations of the best response correspondences for player's 1 and 2. Observe, as expected by Theorem 1.13, all have an odd number of equilibrium situations.

Remark
Suppose player 1 has a pure strategy which is strictly dominated, i.e., player 1 always prefers one pure strategy for all mixed strategies selected by player 2. Then the corresponding best response correspondence is a vertical line at the corresponding point on the axis labelled x_1. Similarly for player 2.

Remark
Any point (a, b) from the space $[0, 1] \times [0, 1]$ (as portrayed in Figure 1.6-1.9) completely describes a mixed strategy for both players where $x_1 = a$ and $x_2 = b$.

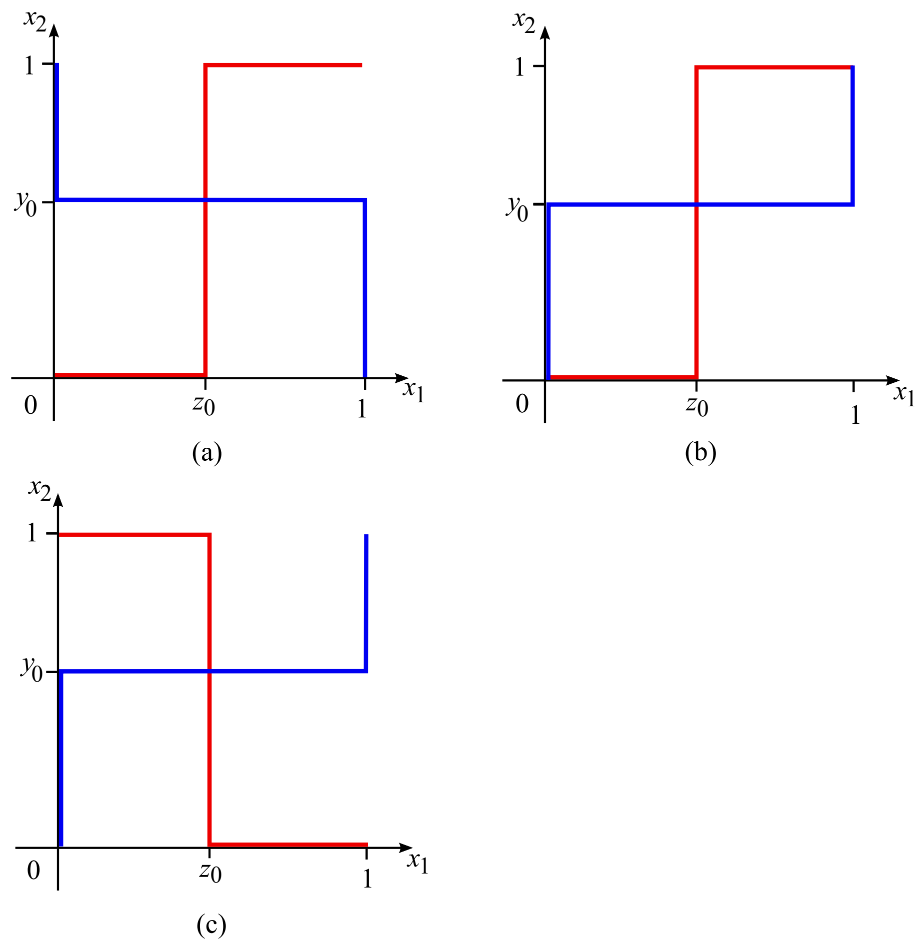


Figure 1.9: Remaining configurations of the intersetion of best response correspondences: (a) and (c) have 1 equilibrium point and (b) has 3 equilibrium points.

(End Example)

1.4 Degenerate Games

In addition to identifying equilibrium situations, properties of the best response correspondences also provide clarification into the distinction between degenerate and non-degenerate games. Returning to Example 1.15 assume the payoff function for player 1 remains as given in (1.30). Then alter player 2's payoff function from (1.31) such that his optimal strategy when $x_1 = 1$ is a totally mixed strategy and for all other situations of the game player 2's optimal strategy is s_2^1. In Figure 1.10, (a) shows the new restricted payoff functions for player 2 and (b) is the resulting intersection of both player's best response correspondences.

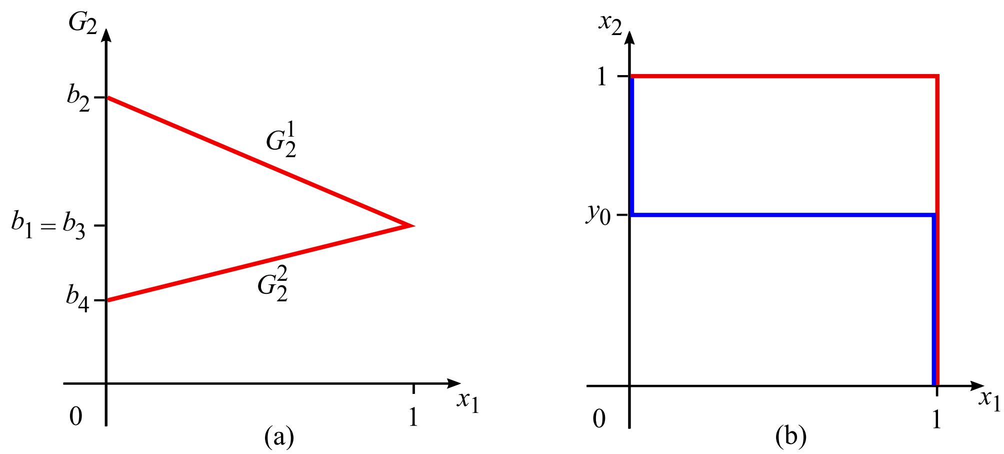


Figure 1.10: (a) the restricted payoff functions for palyer 2 and (b) the intersection of the best response correspondences for player's 1 and 2.

In (b) of Figure 1.10 there are two points of intersection and by Theorem 1.13 this game is degenerate. The first is $(0, 1)$ and corresponds to a situation which is completely determined; that is the probabilities assigned to x_1 and x_2 are unique. The second is an intersection of dimension 1 and occurs when $x_1 = 0$ and for all $x_2 \in [0, y_0]$. Observe at this 1-dimensional intersection the tangents of both A_1 and A_2 coincide.

For bimatrix games the definition of degeneracy is well defined and numerous (equivalent) definitions exist (see [von Stengel, 2002] for example). A popular definition of non-degeneracy for these games is as follows and can be found in [von Stengel, 2002].

Definition 1.16 (Non-Degenerate and Degenerate Bimatrix Games)
A bimatrix game is called non-degenerate *if the number of pure best responses to a mixed strategy never exceeds the size of its support. A bimatrix game is then* degenerate *if this condition fails for at least one situation of the game.*

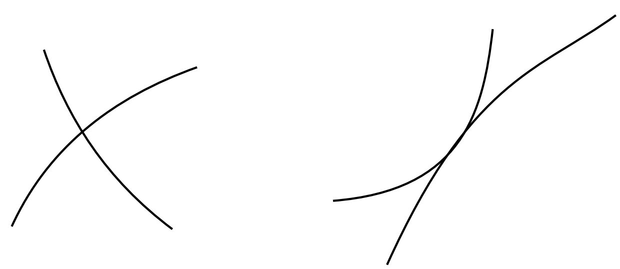
Unfortunately the notion of degeneracy becomes increasingly more complicated as the size of a game increases and currently there is no formal or precise definition for games with more than 2 players. This is not a question we aim to answer in this Thesis so instead we make use the definition presented by Rosenmüller in his proof of Nash's Theorem for N-person games [Rosenmüller, 1971] (this proof is discussed in Section 1.7).

Definition 1.17 (Non-Degenerate and Degenerate N-Player Games)
A game is said to be non-degenerate *if each segment of all best response correspondences is a smooth manifold and all intersections of such correspondences are transverse.*

Remark
By transverse we mean the intersection of an additional best response correspondence reduces the dimension of the intersection by 1. Additionally each intersection must be a smooth manifold.

In particular, in a non-degenerate game, at the point of intersection the linear approximation, or tangent, of each best response correspondence does not coincide with the tangent of any other best response correspondence.

Example 1.18
In Figure 1.11 the intersection seen in (a) is transverse while the intersection in (b) is not.



(a) (b)

Figure 1.11: Tranversal and non-transversal intersections

(End Example)

Observe a slight alteration to strategy choice or payoff function for player i results in a small perturbation to the best response correspondence A_i. Then, in the non-degenerate game, such changes will not affect the transversal property of these intersections. Consequently non-degenerate games are 'stable' and account for almost all games. This is in contrast to the degenerate case where in this scenario a slight perturbation within the game will cause the positioning of the best response correspondences to alter enough to satisfy the conditions to be non-degenerate.

Remark
The condition of non-degeneracy also extends to the intersection of $k \leq N$ best response correspondences.

1.5 Extensive Form

The **extensive form** representation of a game displays all relevant information using a tree. Within this tree each node represents a position of the game, the edges are the strategy decisions made by the player's and the end nodes give the payoff values. Consider a two player dyadic game where player 1 can choose between strategies s_1^1 and s_1^2 and player 2 between s_2^1 and s_2^2. Let $p_1 = (x_1, x_2) \in P_1$ and $p_2 = (y_1, y_2) \in P_2$. The first node represents the beginning of the game and without loss of generality we assume player 1 selects his pure strategy first. He has two choices to make, he can either select strategy s_1^1 or s_1^2 and to represent this choice there are two edges leading from the first node. These edges then terminate with a second node which represent player 2's choice of strategy, and once again two edges lead from each node. The final nodes give the payoffs assigned to each player. Figure 1.12 represents this simple game.

An important definition connected to extensive form games is that of an **information set**. This is a set of nodes associated to one player for a single position of the game (i.e., the number of moves before each node in the set, for all players, is the same). The distinguishing feature of the information set is that, given the information the player currently has, he cannot distinguish between the different nodes at the time he makes his move. Therefore at each node he will make the same choice of strategy.

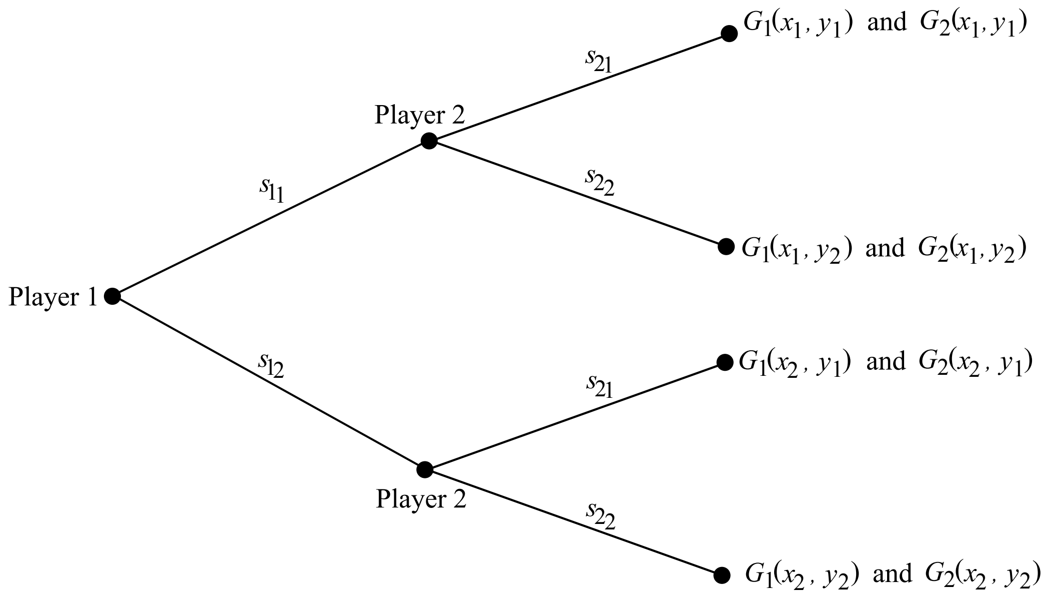


Figure 1.12: Example of a 2-player dyadic game in extensive form

Remark

While this is an important representation of a game, this form will not be considered in the original work of this Thesis.

1.6 Game Theory in Practice

This section provides some examples to highlight the diverse range of applications of game theory.

Military

The Cuban Missile Crisis of 1962 can be simplified as a two player game where each player can choose from 2 pure strategies [Brams, 2001]. This crisis was the result of the US observing the then Soviet Union building nuclear missile sites in Cuba. The Soviet Union needed to decide if they would remove (R) their missiles or maintain them (M) while the US considered both a blockade (B) and air strike (A) to remove these missile bases .

For this simplified representation of the situation the pure strategy set for the US was $\{A, B\}$ and for the Soviet Union was $\{M, R\}$. This generated four different outcomes to which each side could assign a number from 1 to 4, where 4 represents the most favourable outcome. Figure 1.13 shows the corresponding payoff table.

	A	B
M	(1,1)	(4,2)
R	(2,4)	(3,3)

Figure 1.13: Payoff table for Cuban Missile Crisis [Brams, 2001]

The situation (A, M) with payoff (1,1) represents the outcome of a nuclear war while the situation (B, R) with payoff $(3, 3)$ represents a compromise. Neither of these situations are equilibrium points of the corresponding 'game' as at least one side can achieve a more favourable outcome by altering their strategy. This leaves the situations (B, M) with payoff $(2, 4)$ where the Soviet Union claim victory and the US are defeated and the reverse situation of (A, R) with payoff $(4, 2)$ as the pure strategy equilibrium situations. Observe the situations representing Nash equilibria are not the ones representing compromise.

We now examine a more complex model, the relationship between China and Taiwan. The BBC News Country Profile Report [BBC News, 2008] describes Taiwan as being practically independent of China. However this is a situation China would like to change and, if necessary, is prepared to use force to ensure Taiwan is reunited with the mainland. Washington, USA, is the main weapons supplier to Taiwan, and this has contributed in part to the offset of military threat. As such it appears the largest and most significant risk of military conflict between China and USA would involve Taiwan. This is discussed and examined using game theory in 'A Game Theory View of Military Conflict in the Taiwan Strait' by Frank and Melese [Frank and Melese, 2003]. The paper analyses the situation and represents it as a game in extensive form; a summary of this can be seen in Figure 1.14.

The end node "US opts to defend" extends to a secondary graph (not given but included in the referenced paper) which includes the different scenarios resulting from US and China taking pre-emptive measures and deciding to strike first. The term blockade refers to China taking the decision to prevent supplies reaching Taiwan. Therefore the outcome of all possible scenarios can be effectively modelled as an extensive form game and thus can be analysed as such to enable each side to arrive at the most optimal outcome. According to the paper [Frank and Melese, 2003] the action taken by the US will depend on the 'mood' of China. In particular if China is patient rather than impatient, and provided the likelihood of China succeeding is slim, it will be likely military conflict can be avoided. It is in analysing these factors along with costs, economy and political realities which will

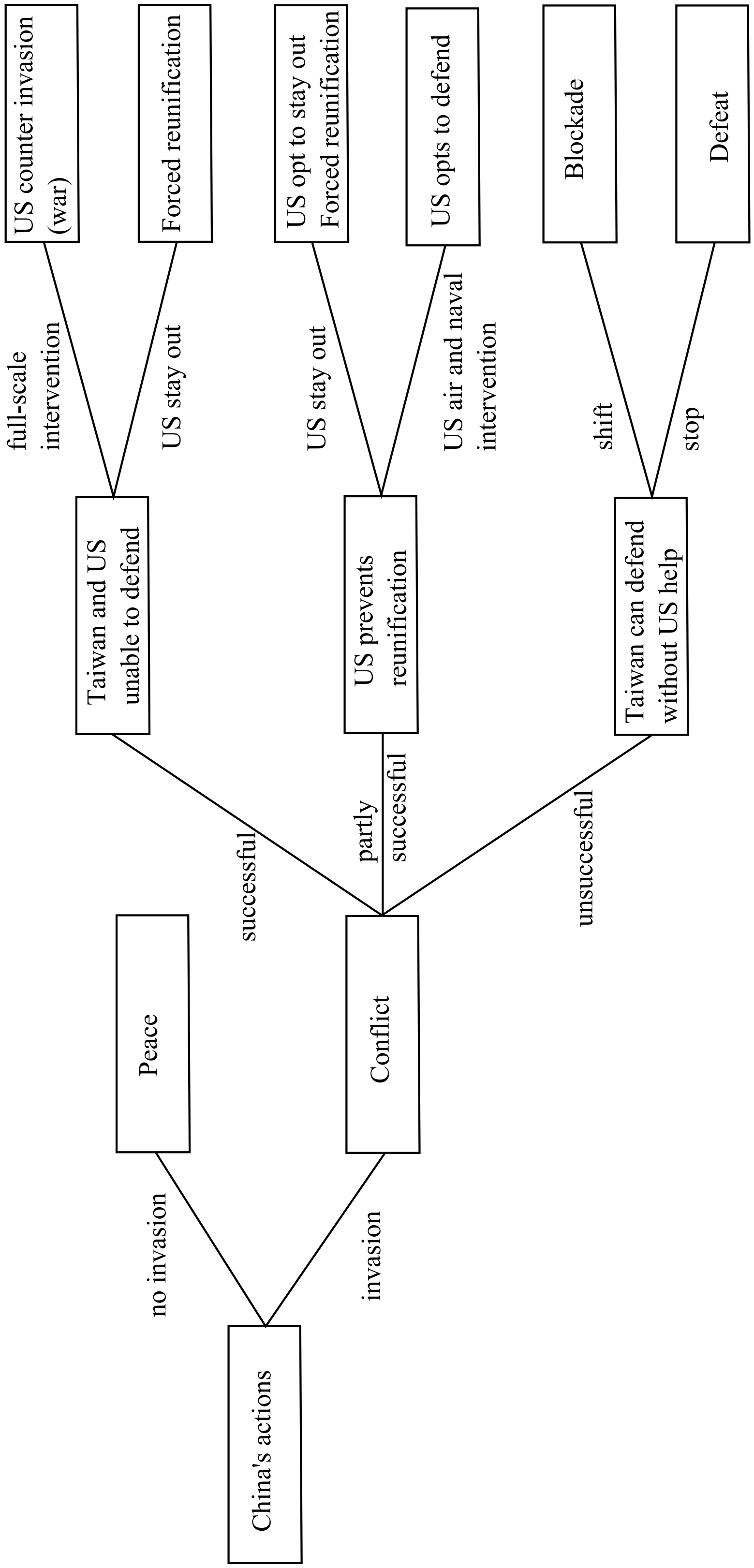


Figure 1.14: Summary of extensive form game representing the risk of military conflict between USA and China, adapted from [Frank and Melese, 2003]

realise a solution. In particular a blockade may prove to be too costly to US/Taiwan and so in the situation where either a blockade of force reunification exists choosing the latter may turn out to be the best solution.

Economics and Cournot's Model of Oligopoly

Cournot's Model of Oligopoly [Cournot, 1838] was developed in the 19th Century, long before game theory was first formalised, and models the interactions of firms competing for business. Suppose there are N firms competing for consumers of a single product then following Chapter 3.1 of "An Introduction to Game Theory" [Osborne, 2004] assume firm i spends $C_i(q_i)$ producing q_i units of the product. However producing more output will require a larger expenditure therefore C_i is an increasing function for all $i = \{1, \ldots, N\}$. Each unit is sold at a single price which is influenced by the amount produced by all firms and consumer demand. Let $Z(Q)$ be the market price of each unit where $Q = q_1 + \cdots + q_N$. As the total output Q increases it is assumed the price per unit will fall, therefore while Z is positive it is a decreasing function. Then the revenue generated by each firm is given by $q_i.Z(q)$. Let π_i be a function representing profit for player i then

$$\pi_i(q_1, \ldots, q_N) = q_i.Z(Q) - C_i(q_i) \tag{1.34}$$

This can be expressed in traditional game theory terminology by taking each firm as a player, the set of strategies as the amount of the product produced (this will necessarily be non-negative) and each player's payoff will be the profit made.

Politics

In 1929 Hotelling used game theory to model spatial competition [Hotelling, 1929]. Additionally he observed his work could be applied to fields as diverse as politics and in particular could be used as a tool to help politicians choose policies to increase their chance of winning the largest proportion of the votes in an election. A simplified example takes the candidates standing for election as the players and each policy as a number from a sliding scale determining how 'left' or 'right' wing it maybe. A strategy is then a selection of these polices and a situation is the set of policies chosen by all candidates. Those eligible to vote select their personal preference from those policies put forward and the candidate who achieves the most votes wins. Each player wants to win, would prefer to win than to tie for first place, but would prefer to tie for first place than to come in any other position. Should a player tie for first place they want to do this with as few other candidates as possible. A payoff function is then defined and is dependent on the position each strategy achieves; i.e., if it wins outright, ties for first place or achieves a different position. This is described in "An Introduction to Game Theory" [Osborne, 2004] (page 71) and relies on each voter choosing the policy closest to their ideal position. Suppose each voter is asked to describes their perfect policy, then this can be given a number corresponding to the same scale as before. Using this information the median value can be determined and will define the *median policy*. The equilibrium point of this scenario results in each candidate selecting policies which lie as close as possible to this median policy. In his paper Hotelling observed this trend in the US elections, where the policies chosen by the Republican and Democratic parties are usually similar.

Predator-Prey

Chen, Bao and Yan's article "A Predation Behavior Model Based on Game Theory" [Chen et al., 2005] uses game theory to model the behaviour of predators. A predator can either catch his prey by running fast or he can wait and remain still so the prey will come to him. The prey can avoid being caught by either out running the predator or remain hidden so the predator cannot sense he is there. The payoff given in the article is an algebraic sum of energy income and expenditure and consequently relates directly to the size of the animals under consideration. It is assumed this can be modelled using a zero-sum game and neither prey or predator will know the tactics of the other.

The payoff matrix for the predator is given in table in Figure 1.15

	Prey Runs	Prey Hides
Predator Runs	e_1	e_2
Predator Hides	e_3	e_4

Figure 1.15: Payoff matrix for predator behaviour, taken from [Chen et al., 2005]

Where e_1, e_2, e_3, e_4 is the notation used in [Chen et al., 2005] to define the solutions given by the payoff sum. Their result coincides with observations made in nature with larger animals choosing to chase their prey rather than trying to hide while smaller animals are better at conserving energy and hiding to avoid their predators. This analysis assumes animals are able to select different strategies for different behaviour, but which there is no evidence to support this. However the authors believe

> "There must be an optimum solution behind any biological behavior."
> [Chen et al., 2005]

1.7 Proving Nash's Theorem

In section 1.6 the example given for an application of game theory in economics details Cournot's Model of oligopoly which describes the interactions of firms competing for business. In addition to defining this model, Cournot also describes a method to find its solution. This method was first devised and used in the 19th Century, long before any formalisation of game theory. However when describing Cournot's model in terms of game theory it becomes apparent its solution set corresponds exactly to the equilibrium situations of the game. While Nash's Theorem extends Cournot's work to the formal description of game theory, even in 1950, Nash's work was not completely original. In 1947 von Neumann and Morgenstern provided a proof of the Minimax Theorem for 2 player zero-sum games [von Neumann and Morgenstern, 1944]. Therefore John Nash's achievement was to be the first mathematician to provide a proof of Theorem 1.12 for all non-cooperative games with a finite number of players, each with a finite number of pure strategies. We detail this proof and Nash's subsequent improvement to it in 1951. This is followed by a chronological overview of other proofs for this Theorem and in particular to the proof of the extension of Nash's Theorem, Theorem 1.13.

1.7.1 John Nash - 1950 & 1951

John Nash provided two proofs of his famous result. His first result was presented in 1950 [Nash, 1950b] and makes use of Kakutani's Theorem [Kakutani, 1941]: a generalisation of Brouwer's fixed point theorem.

Theorem 1.19 (Brouwer's Fixed Point Theorem [Borowski and Borwein, 2002])
Every continuous mapping of a compact convex set into itself has a fixed point.

Kakutani's Theorem generalises this result:

Theorem 1.20 (Kakutani's Fixed Point Theorem [Borowski and Borwein, 2002])
Every correspondence Φ that maps a compact convex subset C of a locally convex space into itself, with a closed graph and convex non-empty images (i.e., $\Phi(x)$ is a non-empty convex set of C for all $x \in C$) has a fixed point, $x \in \Phi(x)$.

Recall in an N-player non-cooperative game Γ, a situation $\mathbf{p}$ from the product space P is a vector containing a single mixed strategy for each player. Then in terms of our notation, the proof Nash presented in [Nash, 1950b] is as follows.

Definition 1.21 (Counters [Nash, 1950b])
A vector $\mathbf{p} \in P$ counters *a second vector $\mathbf{p'} \in P$, if the strategy of each player in $\mathbf{p}$ yields the highest obtainable expectation (payoff) for that player against the $N-1$ strategies of the other player's in $\mathbf{p'}$.*

Observe Nash's definition of counters is equivalent to the set of best responses for a player i to a situation from $P^{(i)}$. Assume $\mathbf{p}$ is given by $(p_1, \ldots, p_N)$ where $p_i \in P_i$ for all $i = \{1, \ldots, N\}$. Then by Definition 1.21 if $\mathbf{p}$ counters itself then, for all $i = \{1, \ldots, N\}$, player i's optimal payoff for situation $\{p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N\}$ is achieved at strategy p_i. This is identical to the definition of an equilibrium point, Definition 1.10. Therefore the set of all self-countering vectors $\mathbf{p}$ is the set of equilibrium points in Γ.

For every $\mathbf{p^{(i)}} \in P^{(i)}$ an optimal strategy for player i can be identified and consequently every vector $\mathbf{p}$ must have a countering set. For all $i = \{1, \ldots, N\}$ let $Z_i \subset S_i$ denote the set of pure best responses for player i for strategy $\mathbf{p^{(i)}}$. Then the set of all best responses is the convex hull of Z_i, which is a convex and closed set. Then the graph of a given vector $\mathbf{p}$ to its countering set is a one-to-many correspondence from the product space $P_1 \times \cdots \times P_N$ into itself. Therefore the conditions of Kakutani's Theorem hold and can be used to prove the existence of a fixed point. These fixed points are the self-countering N-tuples and consequently there is at least one equilibrium point.

In 1951 John Nash game a different proof to his work [Nash, 1951]. By constructing a continuous function with fixed points corresponding once again to the equilibrium situations of the game, Nash was able to use Brouwer's Fixed Point Theorem directly.

Define the game Γ as before. Recall $(\mathbf{p}, p'_i)$ is the situation $\mathbf{p} \in P$ with player i changing his strategy to $p'_i \in P_i$. Similarly for $(\mathbf{p}, s_i^j)$ with player i choosing pure strategy $s_i^j \in S_i$ instead.

Nash's proof requires the following function:

$$\varphi_i^j(\mathbf{p}) \ := \ \max\{0, \ G_i(\mathbf{p}, s_i^j) - G_i(\mathbf{p})\,\} \tag{1.35}$$

Then for each mixed strategy $p_i \in P_i$ a modification p_i' is defined by:

$$p_i' \ := \ \frac{p_i \ + \sum\limits_{S_i} \varphi_{i_j}(\mathbf{p}) s_i^j}{1 \ + \sum\limits_{S_i} \varphi_{i_j}(\mathbf{p})} \tag{1.36}$$

with associated vector $\mathbf{p'} = (p_1', \ldots, p_N')$.

Define $T : \mathbf{p} \mapsto \mathbf{p'}$ to be a continuous mapping. Since the aim is to use Brouwer's Fixed Point Theorem to establish the result, Nash verifies the fixed points of T to be equilibrium situations by considering situation $\mathbf{p}$. At an equilibrium situation no player has any incentive to change their strategy, therefore each player's payoff is at a (local) maximum. In particular if $\mathbf{p}$ is an equilibrium situation, and if player i's pure strategy s_i^j is optimal for $\mathbf{p^{(i)}}$, then $(G_i(\mathbf{p}, s_i^j) - G_i(\mathbf{p})) = 0$ and by definition $\varphi_i^j(\mathbf{p}) = 0$. Alternatively if s_i^j is not maximal then $(G_i(\mathbf{p}, s_i^j) - G_i(\mathbf{p})) < 0$ and once again $\varphi_i^j(\mathbf{p}) = 0$. Therefore in the definition of p_i' the denominator is 1 and the numerator is p_i. This must be true for all players $i = \{1, \ldots, N\}$ and therefore $p_i = p_i'$ for all i.

For the inverse implication consider a fixed point of the function T i.e., a situation $\mathbf{p} \in P$ such that $T(\mathbf{p}) = \mathbf{p}$. Then for all $i = \{1, \ldots, N\}$ it must be the case that $p_i = p_i'$ and in particular, by (1.36), $\varphi_i^j(\mathbf{p}) = 0$ for all $i, j \in \{1, \ldots, l_i\}$. This is equivalent to each player being unable to improve his payoff by changing his strategy choice alone and thus must represent and equilibrium situation.

Finally, since Brouwer's Fixed Point Theorem guarantees the existences of at least one fixed point there is at least one equilibrium situation.

1.7.2 Lemke and Howson - 1964

In 1964 Lemke and Howson [Lemke and Howson Jr, 1964] produced an algebraic proof of Nash's Theorem for 2-player non-degenerate games. This proof allowed Nash's Theorem to include the statement

the number of equilibrium points is finite and odd.

Consider a bimatrix game where player 1 has m pure strategies and player 2 has n. Denote the payoff matrices for each player by A and B with mixed strategy vectors $\mathbf{x_0} \geq 0$ and $\mathbf{y_0} \geq 0$ for player's 1 and 2 respectively. Let e be a column vector with all elements equal to one, and whose order can be understood from the context, then we must have $e^T\mathbf{x_0} = e^T\mathbf{y_0} = 1$.

The equilibrium situations of a game can be identified by solving the appropriate $m+n$ inequalities. In their paper, Lemke and Howson solved this problem by transforming the game into a linear complementarity problem (LCP) (see their paper [Lemke and Howson Jr, 1964] for details). When transforming the standard game theory problem into an LCP two additional variables, which were not described in the preceding paragraph, appear. These variables can be 'removed' via a normalisation procedure. However this simplification to the LCP requires matrices A and B to be strictly positive. If this is not the case then for a matrix E, of the same order as A and B and with all elements equal to 1, and for some constant k, replace A by $A+kE$ and B by $B+kE$ such that both matrices are (strictly) positive. This process does not alter the best responses of the game. Moreover, Lemke and Howson assumed that A and B are costs to be minimized, not profits to be maximised. Therefore A and B are positive **cost matrices**.

The resulting LCP is

$$\begin{aligned} B^T\mathbf{x} - e &\geq 0, \qquad x \geq 0 \qquad && \mathbf{y}^T(B^T\mathbf{x} - e) = 0 \\ & && \text{and} \\ A\mathbf{y} - e &\geq 0, \qquad \mathbf{y} \geq 0 \qquad && \mathbf{x}^T(A\mathbf{y} - e) = 0 \end{aligned} \tag{1.37}$$

Then since (1.37) is equivalent to the conditions which need to be satisfied for a Nash equilibrium situation, any pair of vectors $(\mathbf{x}, \mathbf{y})$ satisfying (1.37) will also be called an equilibrium situation. Observe, while $\mathbf{x}$ and $\mathbf{y}$ are vectors of the same dimension as the mixed strategies $\mathbf{x_0}$ and $\mathbf{y_0}$ they are not probability distributions. The subsequent proof of Nash's Theorem by Lemke and Howson is then equivalent to solving an LCP. They achieve this using geometric considerations, in particular convex polyhedra.

For vector $\mathbf{x}$, associated to player 1, define the convex polyhedron X as:

$$X \;=\; \{\mathbf{x} \,|\, \mathbf{x} \geq 0,\; B^T\mathbf{x} \,-\, e \geq 0\} \tag{1.38}$$

Then points in X are required to satisfy $m + n$ inequalities. Explicitly if matrix $B = (b_1, \ldots, b_n)$ and the $n \times n$ identity matrix $I = (e_1, \ldots, e_n)$ then these inequalities are given by

$$\begin{array}{rcccl} e_i^T\mathbf{x} & \geq & 0 & & i = 1, \ldots, m \\ b_j^T\mathbf{x} - 1 & \geq & 0 & & j = 1, \ldots, n \end{array} \tag{1.39}$$

A face of X is identified when a subset of these inequalities are satisfied as an equality. Equalities occur when either the corresponding component of vector $\mathbf{x}$ is 0 or satisfies the equation $b_j^T\mathbf{x} - 1 = 0$. Reverting back to the game theory interpretation, that is either a given pure strategy does not belong to the player's support, of the situation in question, or it identifies a pure best response for the opponent. Those inequalities which are satisfied as an equality are said to be **binding**.

The definition of non-degeneracy in the Lemke-Howson paper is equivalent to restricting the maximum number of inequalities a given vector $\mathbf{x}$ can satisfy as an equality to m. Any points which satisfy the full m equalities will be called **extreme points**. The non-empty subset of points which satisfy the same $m - 1$ equalities are called **open edges**. Within the set of open edges we can identify a subset of edges with a specific property. These edges are referred to as **unbounded edges** of X and occur when the $m - 1$ equalities satisfied all define components of $\mathbf{x}$ to be 0. Equivalently this corresponds to the vector in Euclidean space which represents a unique pure strategy choice for player 1. However since we are working with vectors $\mathbf{x}$ and not probability distributions $\mathbf{x_0}$ this vector is of infinite length extending away from the polyhedra.

Remark
This coincides with the definition of non-degeneracy we will use this in this Thesis. In particular in the geometric setting an extreme point will correspond to a vertex of the convex polyhedron X while an open edge is a 1-dimension face.

The same is repeated for player 2, with the convex polyhedron Y being defined as:

$$Y \;=\; \{\mathbf{y}\,|\,\mathbf{y} \geq 0,\; A\mathbf{y} \;-\; e \geq 0\} \tag{1.40}$$

Then once again the conditions of non-degeneracy ensures any point can satisfy a maximum of n of the $m+n$ inequalities defined in Y as an equality. Any point which does satisfy this maximum number is extreme.

Define the set $Z = (X, Y)$ to be the cartesian product of the polyhedra X and Y. A point $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ is extreme is both $\mathbf{x}$ and $\mathbf{y}$ are extreme and belongs to an open edge if one of $\mathbf{x}$ or $\mathbf{y}$ is an extreme point and the other lies on open edge. Finally an edge in Z is unbounded if both $\mathbf{x}$ and $\mathbf{y}$ lie on unbounded edges in X and Y respectively. Each open edge in Z has two end points and if two distinct open edges have the same end point then they are called **adjacent**. The proofs of all results and statements which follow can be found in the referenced publication, [Lemke and Howson Jr, 1964].

Referring back to the LCP described in (1.37), a point $z \in Z$ is an equilibrium situation if it satisfies the following set of $(m+n)$ equations:

$$\begin{aligned} (e_i^T\mathbf{x})(a_i^T\mathbf{y} - 1) \;&=\; 0 \qquad i = 1, \ldots, m \\ (e_j^T\mathbf{y})(b_j^T\mathbf{x} - 1) \;&=\; 0 \qquad j = 1, \ldots, n \end{aligned} \tag{1.41}$$

Where matrix $A^T = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$.

If a point z satisfies the conditions given in (1.41) then z necessarily satisfies $m+n$ of the equalities described in (1.38) and (1.40). Consequently any equilibrium situation is also extreme (note the reverse implication does not hold). The proof of Theorem 1.13 is achieved by constructing paths of adjacent open edges through Z which terminate with an extreme point.

For a fixed $r \in \{1, \ldots, m+n\}$, define the set S_r to be the subset of points from Z such that all $(m+n)$ of the equilibrium conditions (1.41), with the possible exception of $(e_r^T \mathbf{y})(b_r^T \mathbf{x} - 1) = 0$, are satisfied. Then the sequence of adjacent open edges of S_r together with their end points, are called r**-paths**. Each point of S_r is either an extreme point of Z or a point on an open edge of Z and further there is precisely 1 unbounded edge of Z composed of points of S_r (this is proved in two Lemmas from [Lemke and Howson Jr, 1964]). Recall, by definition, such an edge has just one end point.

A Lemma from the paper demonstrates if a point $z \in S_r$ is extreme there are one or two open edges consisting entirely of points from S_r which have z as an end point. Further z is an equilibrium point if and only if there is just one such edge.

By the Lemmas given in the referenced paper at least one extreme point of Z belongs to S_r. Select such an extreme point and construct the adjacent r-path through S_r (and hence Z) to which it belongs to. Suppose $\mathbf{z_1}$ is such an extreme point then $\mathbf{z_1}$ belongs to at least one open edge. Traverse this edge then if it is not the unique unbounded edge it terminates with an extreme point $\mathbf{z_2} \neq \mathbf{z_1}$. If $\mathbf{z_2}$ is an equilibrium situation then it belongs to just one open edge of S_r, the edge we have used to arrive to this point, and hence the path terminates. If this is not the case then, since $\mathbf{z_2}$ must be the end point of two open edges in S_r, there must be a unique edge to traverse along. Since the number of extreme nodes in Z (and hence S_r) is finite all such paths must terminate. This may happen in one of the following ways

- The path enters the unique unbounded edge
- The path returns to its starting point $\mathbf{z}_1$
- The path reaches an equilibrium point

If a path terminates with its start point $\mathbf{z}_1$ then the path is said to be **closed** and in particular does not contain an equilibrium point. Now consider those paths which are not closed. Then we may assume $\mathbf{z}_1$ is the beginning point of the path and belongs to just one open edge. In particular $\mathbf{z}_1$ is an equilibrium point. Observe these paths cannot be circular as this would result in some $\mathbf{z}_i$ begin the end of three open edges which is a contradiction. The path then terminates with either a second equilibrium point, disjoint from $\mathbf{z}_1$, or enters the unbounded edge. In the first case the path identifies two equilibrium points and in the second case just one. The paper shows there is a unique unbounded edge in S_r and so in particular there is at least one equilibrium point in the original bimatrix game. Given any other equilibrium point must appear as a pair the total number of such points must be odd.

Finally the Lemke-Howson constructive procedure is reached. Fix a choice r and identify the corresponding unique unbounded edge of S_r. Using this edge as a starting point traverse the resulting r-path as defined above. This path will terminate with a single equilibrium point.

When Γ is a degenerate game the sets X and Y can be perturbed to ensure they satisfy the conditions of non-degeneracy. There are many well known ways to deal with this perturbation but these details go beyond the scope of this Thesis. Further information on this can be found in the original paper Lemke-Howson paper [Lemke and Howson Jr, 1964] or alternatively in [von Stengel, 2002]. However, since in the non-degenerate game we can guarantee the existence of at least one equilibrium point this must also be true for degenerate games.

L.S. Shapley - 1974

The work we will present in this Thesis will build upon the proof of Nash's Theorem provided by Lemke and Howson [Lemke and Howson Jr, 1964]. To allow a more intuitive comparison between our work and theirs we give the graphical description of the Lemke-Howson constructive procedure as presented by Shapley [Shapley, 1974]. We will refer back to Shapley's interpretation when wanting to discuss the Lemke-Howson paper in connection to our work.

Using geometry, Shapley presents the constructive procedure contained in the Lemke-Howson paper in an accessible and visual way. By defining a labelling system for bimatrix games Shapley effectively explains the content of the Lemke-Howson paper.

Let Γ be a non-degenerate bimatrix game where each player $i = \{1, 2\}$ has pure strategy set $S_i = \{s_i^1, \ldots, s_i^{l_i}\}$ and mixed strategies $p_i = (x_i^1, \ldots, x_i^{l_i})$. Let P_1 and P_2 denote the set of all mixed strategies and A and B represent the payoff matrices for player's 1 and

2 respectively. Then, for $i = \{1, 2\}$, recall P_i is a simplex of dimension $l_i - 1$. For ease of notation assume each pure strategy can be represented by a unique natural number; in particular let $S_1 = \{1, \ldots, l_1\}$ and $S_2 = \{l_1 + 1, \ldots, l_1 + l_2\}$.

Define the following two sets

$$\begin{aligned}\tilde{P}_1 &= P_1 \cup \left\{ p_1 \geq 0 \,:\, \sum_{i=1}^{l_1} x_1^i \leq 1 \text{ and } \prod_{i=1}^{l_1} x_1^i = 0 \right\} \\ \tilde{P}_2 &= P_2 \cup \left\{ p_2 \geq 0 \,:\, \sum_{j=1}^{l_2} x_2^j \leq 1 \text{ and } \prod_{j=1}^{l_2} x_2^j = 0 \right\}\end{aligned} \tag{1.42}$$

Observe each $\tilde{P}_i$ contains the original mixed strategy simplex P_i plus faces of simplices of higher dimension which have a non-empty intersection with at least one boundary face of P_i. The condition on $\tilde{P}_i$ that the sum of mixed strategies x_i^k must sum to 1 ensures such 'extended simplices' are bounded in the positive orthant by P_i. Within $\tilde{P}_i$, Shapley defines the following convex polyhedra.

$$\begin{aligned}P_i^j &= \{p_i \in \tilde{P}_i \,:\, x_i^j = 0\} \qquad \text{for } j \in S_i \\ P_i^k &= \{p_i \in \tilde{P}_i \,:\, k \in S_{i'} \text{ is a pure best response for player } i' \neq i \text{ against strategy } p_i\}\end{aligned} \tag{1.43}$$

Then the sets P_i^j provide a covering of the space $\tilde{P}_i \setminus P_i$ and since, for every strategy from P_i player $i' \neq i$ must have a pure best response, the sets P_i^k must cover all of P_i. Note for some k we may have $P_i^k = \emptyset$.

Every $p_i \in \tilde{P}_i$ is then assigned the label(s) contained in the set

$$L'(p_i) \;=\; \{q \,:\, p_i \in P_i^q\} \;\neq\; \emptyset \tag{1.44}$$

Note every element in the set P_i^j will be assigned the same label. To complete the description of Shapley's labelling, the pair $(p_1, p_2) \in \tilde{P}_1 \times \tilde{P}_2$ is naturally labelled as

$$L(p_1, p_2) \;=\; L'(p_1) \cup L'(p_2) \tag{1.45}$$

Observe $L'(p_i)$ is a subset of $S_1 \cup S_2$. Then any label from the set $L'(p_i)$ which belongs to S_i details the unplayed strategies selected by player i and those belonging to $S_j, \;\; j \neq i$ indicate the pure best responses for player j. These labels, with interpretation, can be compared directly back to the Lemke-Howson algorithm. In particular the labels contained in $L'(p_i)$ provide the indices of the binding inequalities defined for polyhedra X (if $i = 1$) or Y (if $i = 2$) given in equations (1.38) and (1.40) respectively.

If $L(p_1, p_2) = S_1 \cup S_2 = \{1, \ldots, l_1 + l_2\}$ then the point (p_1, p_2) is **completely labelled**. If $L(p_1, p_2) = (S_1 \cup S_2) \setminus \{r\}$ for some $r \in \{1, \ldots, l_1 + l_2\}$ then the point (p_1, p_2) is **almost-completely labelled**. Using this notation the conditions for a situation $\mathbf{p} = (p_1, p_2)$ to be equilibrium is equivalent to $L(p_1, p_2) = S_1 \cup S_2$, i.e., is completely labelled.

Shapley requires the following three conditions to hold for a bimatrix game Γ to be classified as non-degenerate. For $i = \{1, 2\}$ these are

1. Every non-empty region P_i^j is $(l_i - 1)$ dimensional.
2. The intersection of any two sets of P_i^j is at most $(l_i - 2)$-dimensional
3. No point of $\tilde{P}_i$ can belong to more than l_i of the sets P_i^j.

Remark
These non-degeneracy conditions are equivalent to the intersection of the sets P_i^j being transverse. Consequently this definition of non-degeneracy coincides with the one we shall be using in this Thesis. In particular, since by definition each set P_i^j is a closed convex polyhedra, point 1 from above is equivalent to the sets P_i^j being smooth manifolds. Conditions 2 and 3 then provide the conditions required on the dimension of the intersections. Note if any two of the sets P_i^j intersect with a dimension less than $(l_i - 2)$ then the intersection must involve additional sets. If this was not the case the union of all sets P_i^j would not form a covering of $\tilde{P}_i$.

From the coverings over the space $\tilde{P}_i$ consider the subset of points from $\tilde{P}_i$ which belong to as least $(l_i - 1)$ of the sets P_i^j. An **edge** is then the maximal connected set of points which belong to the same $(l_i - 1)$ sets. Those points which belong to the maximum l_i sets are **nodes**. Then by the non-degeneracy assumptions these are 1-dimensional edges and 0-dimensional vertices. A node shares the same label as the point it represents and, since each point on an edge must be labelled identically, the edges are labelled in the same manner. Denote this graph by F_i, then the cartesian product $F_1 \times F_2$ can be formed. The property of non-degeneracy also allows us to observe at most two nodes can be identified per edge (the end points), such nodes are described as **adjacent**, and each node is the end point of l_i edges.

The set F_i and the labelling procedure described above are illustrated in the following example taken from [Shapley, 1974]

Example 1.22
Let Γ be a non-degenerate bimatrix game where each player has 3 pure strategies. We assume each pure strategy can be assigned a distinct numerical value so $S_1 = \{1, 2, 3\}$ and $S_2 = \{4, 5, 6\}$. We define the payoff matrices as follows

$$A = B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.46}$$

We use this example to demonstrate how to construct the sets $\tilde{P}_1$ and P_1^j. The set P_1 is simply a 2-dimensional simplex. Then, by definition, $\tilde{P}_1$ is constructed from P_1 and the faces of simplices of higher dimension which have a non-empty intersection with at least one boundary face of P_1. Additionally $\tilde{P}_1$ is contained within the non-negative orthant and bounded by P_1. Equivalently $\tilde{P}_1$ is the boundary of the 3-dimensional simplex shown in Figure 1.16

We now discuss the labelling seen in Figure 1.16. First note the simplex P_1 has been constructed over a 3-dimensional axis and not 2-dimensional as seen in Section 1.3.4. Consequently in this representation it is the upper most face which represents P_1 and is labelled as such. The three remaining 2-dimensional simplices of $\tilde{P}_1$ are then labelled by the pure strategy $i \in S_1$ such that the condition $x_1^i = 0$ is satisfied across the entire face. Finally observe the graph of this simplex is simply the vertices and edges seen in Figure 1.16 and consequently is the same object. Note nodes are assigned 3 labels and edges 2.

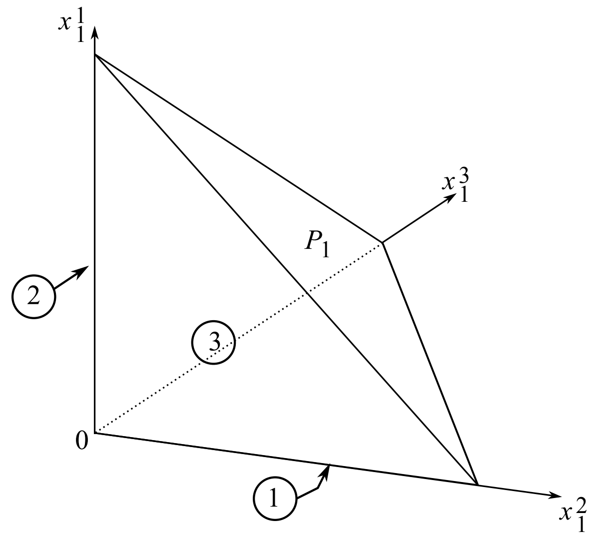


Figure 1.16: The polyhedra $\tilde{P}_1$

Now we are left to determine the sets P_1^k, $k \in S_2$ which forms the covering of P_2. To do this we need matrix B. Reading directly from matrix B we can determine

pure strategy 4 is player 2's best response when player 1 chooses pure strategy 1
pure strategy 5 is player 2's best response when player 1 chooses pure strategy 2
pure strategy 6 is player 2's best response when player 1 chooses pure strategy 3

Player 2 is indifferent between all three of his pure strategies when player 1's mixed strategy is the solution to the following three equations

$$\begin{aligned} x_1^1 + x_1^2 + x_1^3 &= 1 \qquad (1.47) \\ x_1^1 &= x_1^2 \qquad (1.48) \\ x_1^2 &= x_1^3 \qquad (1.49) \end{aligned}$$

The sets P_1^j, for $j = \{4, 5, 6\}$, are defined using the above information and the solutions to the following:

$$x_1^1 + x_1^2 + x_1^3 \quad = \quad 1 \tag{1.50}$$

and

$$x_1^1 \geq x_1^2 \quad \text{and} \quad x_1^1 > x_1^3 \tag{1.51}$$

or

$$x_1^2 \geq x_1^3 \quad \text{and} \quad x_1^2 > x_1^1 \tag{1.52}$$

or

$$x_1^3 \geq x_1^1 \quad \text{and} \quad x_1^3 > x_1^2 \tag{1.53}$$

where when the inequalities are satisfied as an equality player 2 has two pure best responses and just one when a strict inequality holds.

Then Figure 1.17 shows the graph F_1 with labelled sets P_1^k for $k = \{1, \ldots, 6\}$.

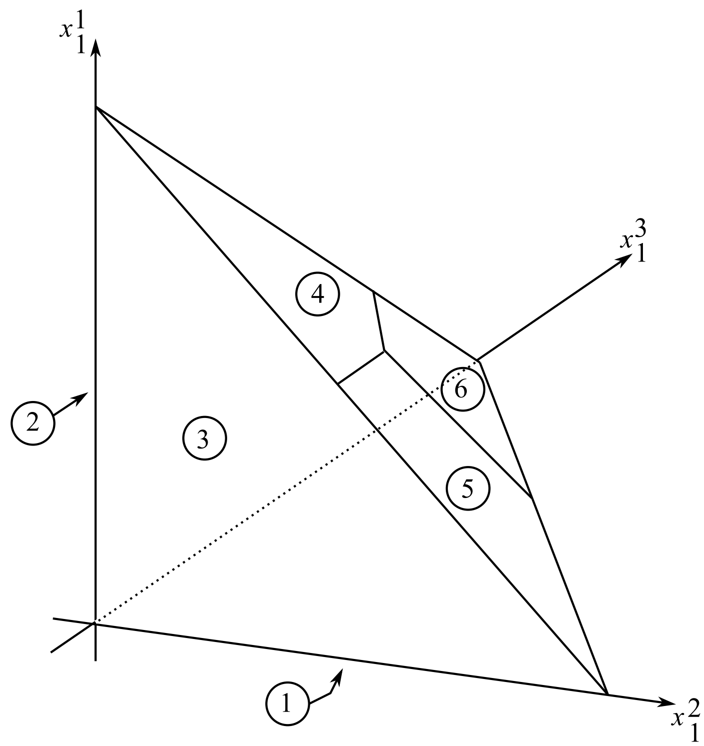


Figure 1.17: The polyhedra F_1 with labelled sets P_1^j

For illustration, the labels at point A are (234), point B are (456) and point C are (135).

Remark
The actual matrix A has not been required to construct $\tilde{P}_1$.

(End Example)

Observe the vertex 0 in Figure 1.17 is labelled (123). It is clear to see such a point will also exist in $\tilde{P}_2$ this time labelled by (456). Therefore the point $(0, 0) \in F_1 \times F_2$ is completely labelled but since it is not defined within $P_1 \times P_2$ it is not an equilibrium point of the game. This is a consequence of the transformation of the convex polyhedra described in [Lemke and Howson Jr, 1964] into a bounded polytope. The point $(0, 0)$ is called the **artificial equilibrium point**.

We now begin to describe the constructive procedure. Intuitively, within the product graph $F_1 \times F_2$, adjacent paths of nodes are followed which terminate with an equilibrium point, or completely labelled node.

Let θ^r be the set of all points $(p_1, p_2) \in \tilde{P}_1 \times \tilde{P}_2$ which are contained in at least $l_1 + l_2 - 1$ sets from $P_1^j \cup P_2^k$ such that $L(p_1, p_2) \supseteq \{1, \ldots, r-1, r+1, \ldots, l_1+l_1\}$. Then θ^r contains the almost completely labelled nodes and edges, with respect to label r, in $F_1 \times F_2$. Similarly let θ contain all completely labelled nodes in $F_1 \times F_2$. Note this set will contain all equilibrium points including the artificial equilibrium point (0,0). Observe $\theta \subset \theta^r$.

Replicating results from the Lemke-Howson paper, Shapley gives the following Lemma

Lemma 1.23
For fixed $r \in S_1 \cup S_2$ every node in θ is adjacent to exactly one member of θ^r and each member of $\theta^r \setminus \theta$ is adjacent to exactly two members of θ^r.

By Lemma 1.23 any node from θ^r belongs to one or two edges of θ^r and exactly one if and only if the node belongs to θ. Starting with a node of θ^r construct the maximal path of adjacent nodes, then this path will take one of two forms. Either the path returns to its starting point or it must terminate with a node from θ. In the first case no completely labelled node can be encountered. In the second case the path must have two end points and thus contain two completely labelled points. If this was not the case then there must be a node of θ^r which belongs to three edges of θ^r which is a contradiction. Since $\theta \subset \theta^r$ this procedure determines there must be an even number of completely labelled nodes in $F_1 \times F_2$. However one of these nodes is the artificial equilibrium given by $(0, 0)$ which proves the Lemke and Howson theorem that every non-degenerate bimatrix game has an odd number of equilibrium points. By starting with the artificial equilibrium $(0,0)$ the path of adjacent nodes, for a arbitrary choice r, must successfully terminate with an equilibrium point. This procedure is illustrated with the following example taken from [Shapley, 1974]

Example 1.24
Let Γ be a bimatrix game where each player has 3 pure strategies and the payoff matrices are as follows:

$$A \;=\; B^T \;=\; \begin{pmatrix} 0 & 3 & 0 \\ 2 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \tag{1.54}$$

Then the graphs F_1 and F_2, relating to $\tilde{P}_1$ and $\tilde{P}_2$, with sets P_1^j and P_2^k are as shown in Figure 1.18.

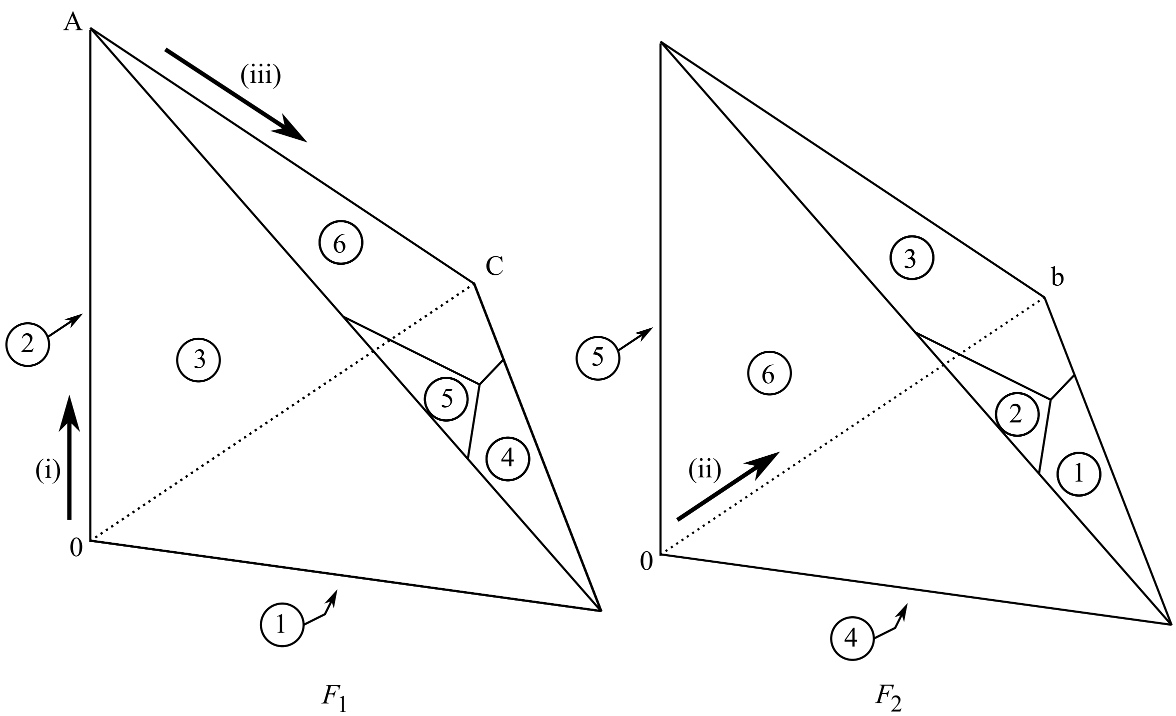


Figure 1.18: Graphs F_1 and F_2 for Γ

Begin the procedure at point $(0,0)$. Without loss of generality we will consider the path belonging to θ^1. In particular we traverse the edge leaving the vertex $(0,0)$ missing label 1. The end point of this path is marked A in F_1 which is labelled by $(236) \cup (456)$. Label 6 is now repeated so to continue following the path belonging to θ^1 traverse the edge leaving the vertex $(A,0)$ missing label 6, found in F_2, which takes us to vertex (A,b) with label $(236) \cup (345)$. Note this choice is unique as there are just two edges which satisfy this property but one was used as the 'arrival' edge. Drop label 3 and progress to vertex (C,b) with label $(126) \cup (345)$ and we have reached a completely labelled node and hence equilibrium point. This path is indicated by the arrows in Figure 1.18, each labelled by the step they represent in the procedure.

(End Example)

Observe the difference here between the constructive procedure by Lemke and Howson and that by Shapley. In particular the paths defined by Lemke-Howson require the identification of an unbounded edge and traversing paths of a polyhedra. Shapley on the other hand transforms the polyhedra defined by Lemke and Howson into a (bounded) polytope containing an artificial equilibrium point.

1.7.3 Rosenmüller and Wilson - 1971

In July 1971 two papers, one by Rosenmüller [Rosenmüller, 1971] and one by Wilson [Wilson, 1971], appeared independently in the *SIAM Journal on Applied Mathematics*. The two authors had independently extended the result of Lemke and Howson [Lemke and Howson Jr, 1964] to N-player non-cooperative games. Their descriptions are included here as a point of interest in relation to Lemke-Howson.

Rosenmüller

Using the same notation as before let Γ represent the N-player non-cooperative, non-degenerate game. Define T_i to be a strict subset of S_i, such that every $s_i^j \in T_i$ is to be assigned the probability zero in a mixed strategy situation. Then $T_1 \cup \cdots \cup T_N$ is the set of all pure strategies which are not to contribute to the support of mixed strategy situation $\mathbf{p} \in P$. Denote by $X_{T_1,\ldots,T_N}$ the subset of mixed strategy situations from P such that for all $i = \{1,\ldots,N\}$ every $s_i^j \in T_i$ is selected with probability zero.

For pure strategy $s_i^j \in S_i$ let $K_i^{i_j}$ be the subset of strategies from $P^{(i)}$ such that strategy s_i^j is optimal for player i. Then for $Y_i \subset S_i$ let $K_i^{Y_i}$ be the subset of strategies from $P^{(i)}$ such that all strategies in Y_i are optimal for player i. Observe $K_i^{Y_i} = \bigcap_{s_i^j \in Y_i} K_i^{i_j}$ and the set $K^{Y_1,\ldots,Y_N}$ combines this information for all players. In particular $K^{Y_1,\ldots,Y_N}$ is the subset of strategies from $\mathbf{p}$ such that every $s_i^j \in Y_i$ is optimal across $\mathbf{p^{(i)}}$ for all $i = \{1, \ldots, N\}$.

Finally define $A_{T_1,\ldots,T_N}^{Y_1,\ldots,Y_N} = A_T^Y$ to be the set of strategies from P which belong to both $K^{Y_1,\ldots,Y_N}$ and $X_{T_1,\ldots,T_N}$. In particular

$$A_{T_1,\ldots,T_N}^{Y_1,\ldots,Y_N} \quad = \quad A_T^Y \quad = \quad K^{Y_1,\ldots,Y_N} \cap X_{T_1,\ldots,T_N} \tag{1.55}$$

If $A_T^Y \neq \emptyset$ then $\mathbf{p} \in A_T^Y$ satisfies the following sets of a equations

1. $G_i(\mathbf{p}, s_i^j) - \lambda^i = 0 \qquad\qquad \forall\ s_i^j \in Y_i,\ \ i = \{1, \ldots, N\}$

   for some λ^i.

2. $x_i^j = 0 \qquad\qquad \forall\ s_i^j \in T_i, \quad i = \{1, \ldots, N\}.$

   i.e., all pure strategies in the set T_i are selected with probability zero.

3. $ep_i = 1 \qquad\qquad \forall\ i = \{1, \ldots, N\} \qquad e = (1, \ldots, 1).$

   i.e., all probabilities assigned to strategies for player i satisfy the conditions of a probability distribution.

For arbitrary T and Y if $\dim A_T^Y = 0$ then A_T^Y consists of finitely many 0-dimensional points called *polyhedra extreme points* or *PEP*'s. Alternatively if $\dim A_T^Y = 1$ then A_T^Y is formed from finitely many smooth curves called *polyhedra edges* or *PE*'s.

Lemma 1.1 from the paper tells us each PEP is the intersection of exactly

$$n - \sum_{|Y_i|=1} 1 \tag{1.56}$$

edges. Where, when $Y_i \geq 2$ each edge is constructed by removing a strategy from either T_i or Y_i (the condition on Y_i is required as it is assumed $Y_i \neq \emptyset$) and is referred to as "the edge $A^{Y_1,\ldots,Y_i-\{i_j\},\ldots,Y_N}_{T_1,\ldots,T_N}$ at x" for a situation $x \in A^Y_T$. Lemma 1.2 determines if a given x is both a PEP and a point of a PE then it must be that x is the end point of the PE.

Define the set B to contain all points which satisfy the conditions of an equilibrium situation except perhaps with respect to strategy $s_N^{l_N} \in S_N$. Then, from the paper, provided x is not of the form $\hat{x}$ (1.57), Lemma 2.1 shows if x is an equilibrium situation then it must belong to B and satisfy the conditions of a PEP. In particular equilibrium situations are characterised as the PEP's which are end points of just one PE. All other PEP's are the end points of two PE's. Define a sub-game of Γ by fixing one player's strategy to one of his pure strategies.

Using the proof by Lemke and Howson to guarantee an odd number of equilibrium situations in the 2 player game, the proof is completed by induction. By assuming there are an odd number of equilibrium points in the sub-game of Γ a situation x can be defined such that x belongs to B, thus proving B is a non-empty set.

Let $\hat{x}^{(N)}$ define an equilibrium point of the sub-game where player N's strategy is fixed to $e_N^{l_N} = (0, \ldots, 1)$, i.e., player N has fixed his strategy to pure strategy $s_N^{l_N}$. The assumption this sub-game contains an odd number of equilibrium points leads to there being an odd number of situations in Γ of the form

$$\hat{x} = (\hat{x}^{(N)}, e_N^{l_N}) \tag{1.57}$$

These are shown in the paper to be both PEPs and to lie in the set B. Pick one of these points, $\hat{x}$ and fix it. The paper shows if $\hat{x}$ is an equilibrium point then it cannot be the end point of an edge in B and if $\hat{x}$ is not an equilibrium point then it is the end point of a unique edge in B. Assume $\hat{x}$ is not equilibrium then following the unique edge it belongs to must lead to another PEP of B. If this point is an equilibrium situation then it cannot be of the form given in equation (1.57) and the 'arrival' edge is the only edge which contains this equilibrium point and the path terminates. If not then the end point is either another PEP of the form of equation (1.57) (and the path terminates) or is another PEP in B and there is a unique departure edge which we can follow and build up a path called a B-path. Since no PEP is the end point of more than two edges, and the original start point $\hat{x}$ is the end point of a unique PEP, the B-path cannot meet itself. Further since the number of PEP's and PE's is finite all such paths can be guaranteed to terminate. Paths must either terminate in an equilibrium point or another PEP of the form given in (1.57). If it ends in a PEP then the path B contains two of the PEP's of the form given in (1.57). However there are an odd number of these points and consequently there must be an odd number of paths which end in an equilibrium point. (This may indeed be the original start point $\hat{x}$).

This procedure may not have accounted for all PEP's from A_T^Y. Since the end points of the B-paths are equilibrium situations any additional equilibrium point must occur in pairs. Thus the total number of equilibrium points in Γ is finite and odd.

Once again this proof provides a constructive procedure to identify an equilibrium situation from Γ.

Wilson

Wilson's proof [Wilson, 1971] follows a similar form to that of Rosenumüller. Z^* is defined to be a closed set of situations from P which partially satisfy the condition to be equilibrium. Then Z is defined to be the subset of Z^* containing the situations from P which satisfy the conditions to be an equilibrium situation except with regards to one player's pure strategy. In particular let $Z(i, i_j)$ denote the set of situations from Z for which player i's pure strategy s_i^j is allowed to fail the conditions of an equilibrium situation.

Those points in $Z(i, i_j)$ which are equilibrium situations are called *complementary* and those which are not are defined to be (i, i_j)-*almost complementary* or just *almost complementary*. Then all equilibrium points are complementary situations of Z. The set Z can be represented geometrically and as such nodes and arcs can be identified. The paper also defines a finite set of boundary conditions for Z. Nodes correspond to situations in Z which satisfy $K = \sum_{i=1}^{N} |S_i|$ boundary conditions, where K is the maximum number any point can satisfy, and are called *extreme* points of Z^*. The arcs of Z are those situations in Z which satisfy $(K - 1)$ of the boundary conditions.

Within Z there are a finite number nodes and thus arcs. Those nodes/arcs which satisfy the definition of complementary are called *complementary nodes/arcs* respectively. The same is true for the definition of (i, i_j)-almost complementary. Then for a given node from the subset $Z(i, i_j)$, Lemma 1 from the paper proves it is an end point of either one or two (i, i_j)-almost complementary arcs and is the end point of one if and only if the node is complementary. Define a (i, i_j)-path to be the maximal connected set of (i, i_j)-almost complementary nodes and arcs. Therefore, since the number of nodes is finite, the set $Z(i, i_j)$ is a the union of a finite number of (i, i_j)-paths.

Like the paper by Rosenmüller [Rosenmüller, 1971], Wilson's proof relies on induction and assumes in the sub-game, as described before, there are an odd number of equilibrium situations. Let $\Gamma(i, i_j)$ represent the $N-1$ player game where player i fixes his strategy to pure strategy s_i^j. Then Lemma 2 from [Wilson, 1971] states an equilibrium situation from $\Gamma(i, i_j)$ corresponds to an (i, i_j)-almost complementary node in $Z(i, i_j)$. Nodes which are defined in such a way are called *initial nodes* and by Lemma 3 of [Wilson, 1971] are the endpoints to precisely one unbounded arc; where an unbounded arc has just 1 endpoint. (Those arcs from $Z(i, i_j)$ which start from any other node have 2 end points and are said to be bounded.) Therefore the assumption $\Gamma(i, i_j)$ has an odd number of equilibrium situations results in there being an odd number of initial nodes in $Z(i, i_j)$.

Choose an arbitrary node $\mathbf{z}$ in $Z(i, i_j)$. The following table highlights the number of almost complementary arcs for which $\mathbf{z}$ is an end point.

Type of Node	**Complementary Arcs**
complementary	one bounded arc
almost complementary	two bounded arcs
initial node	one bounded and one unbounded arc

Traversing a bounded arc will lead to a second node $\mathbf{z^1}$ which will satisfy one of the following conditions

1. $\mathbf{z^1}$ is a complementary node and hence there are no exit arcs
2. $\mathbf{z^1}$ is almost complementary and there is a unique bounded arc to exit along
3. $\mathbf{z^1}$ is an initial node and the only exit path is along an unbounded arc

There are a finite number of nodes in $Z(i, i_j)$ and therefore the path must terminate with one of the above options. If it terminates with option two then $\mathbf{z}^1 = \mathbf{z}$ and the path is circular. In particular the path does not contain any complementary nodes. If a path terminates with one of the remaining options then the number of initial nodes encountered needs to be considered. Since any point is the end point of at most two almost-complementary arcs the maximum number of initial nodes in any one path is necessarily 2. In the case when the two end points of the path are unbounded the path once again contains no complementary nodes. In the case where one end point is an unbounded arc the second must be a complementary node and in the final case where neither end point is an unbounded arc then both end points are complementary nodes. Given there are an odd number of initial nodes, there must be an odd number of paths containing just one unbounded arc, thus producing an odd number of complementary nodes. The case where neither end point is an unbounded arc contributes an even number of complementary nodes. Since complementary nodes are equilibrium points it follows there are a finite and odd number of equilibrium points in Γ.

Remark
Since every equilibrium situation is a complementary node it will be contained in every $Z(i, i_j)$. Therefore the choice of i and $s_i^j \in S_i$ for the set $Z(i, i_j)$ is arbitrary.

Once again this proof naturally allows for a constructive procedure to be defined to identify an equilibrium situation from the game Γ.

Remark
It is noticed the proof by Rosenmüller and Wilson are very similar. In fact it is just in the presentation and definition of the original geometric object which is different. The resulting constructive procedure then builds upon that presented by Lemke and Howson.

1.7.4 Harsanyi - 1973

In 1973 J.C. Harsanyi [Harsanyi, 1973] offered a different approach to the proof of Theorem 1.13. Instead of tracing a path through some geometric object which had been defined directly as a result of situations from P meeting certain criteria of an equilibrium situation, he defined a new class of payoff functions and used these properties to achieve the result. The new functions were logarithmic functions and as such Harsanyi introduced the concept of a *logarithmic game*.

Remark
This proof will not contribute to our work. However we feel it is of interest as it provides a different approach to proving Nash's Theorem. Additionally we want to acknowledge its existence as Harsanyi shared the Nobel prize with Nash (and Selton) in 1994 for their work on Nash equilibria.

For a N-player game Γ, the logarithmic game Λ is defined with the same pure strategy and mixed strategy set as Γ but replaces payoff function G_i with L_i. For all $i = \{1, \dots, N\}$ the function L_i for mixed strategy situations $\mathbf{p}$ is defined to be:

$$L_i(\mathbf{p}) \quad = \quad L_i(p_i) \quad = \quad \sum_{j=1}^{l_i} \log\, x_i^j \tag{1.58}$$

Where x_i^j is the probability player i selects pure strategy $s_i^j \in S_i$. Therefore the payoff for each player in Λ depends only on the individual player. Extending this, Harsanyi defines a one-parameter family of games $\{\Lambda^*(t)\}$ for $0 \leq t \leq 1$ such that for a fixed parameter t the payoff function for player $i = \{1, \ldots, N\}$ is

$$L_i^*(\mathbf{p}, t) \quad = \quad (1-t)G_i(\mathbf{p}) \; + \; tL_i(p_i) \tag{1.59}$$

Then $\Lambda^*(0) = \Gamma$ and $\Lambda^*(1) = \Lambda$. For $0 < t \leq 1$ the game $\Lambda^*(t)$ is said to be a *logarithmic game* with Γ being the *original game* and Λ the *pure logarithmic game*. For Γ, the paper introduces the following definitions

- An equilibrium point $\mathbf{p} = (p_1, \ldots, p_N)$ is **strong** if all components of $\mathbf{p}$ satisfy equation (1.19) with strict inequality for all $i = \{1, \ldots, N\}$ and p_i, $p_i' \in P_i$ where $p_i \neq p_i'$.
- An equilibrium point is **weak** if it is not strong
- If Y_i is the set of pure strategies which are assigned a positive probability in p_i then an equilibrium point $\mathbf{p}$ is said to be **quasi-strong** if player i has no pure strategy best reply to $\mathbf{p}^{(\mathbf{i})}$ which is not contained in Y_i.
- An equilibrium point that is not quasi-strong is **extra weak**
- Γ is **quasi-strong** if *all* equilibrium points are quasi-strong and will be **extra weak** if at least one equilibrium is.

It is known the best response for each player i in the original game $\Gamma = \Lambda^*(0)$ maybe a pure or mixed strategy. In contrast, for $0 < t \leq 1$, the game $\Lambda^*(t)$ can only have a totally mixed strategy as a best reply; if this was not the case then for at least one strategy we have $x_i^j = 0$ resulting in $L_i^* = -\infty$.

For $i = \{1, \ldots, N\}$ and $k = 2, \ldots l_i$ the equilibrium points of the logarithmic game can be characterised by equations of the form

$$(1-t)x_i^1 x_i^k \left[G_i(\mathbf{p}, s_i^k) - G_i(\mathbf{p}, s_i^1)\right] \;+\; t(x_i^1 - x_i^k) \;\;=\;\; 0 \tag{1.60}$$

Let K^* be the number of equations of the form (1.60) then $K^* = \sum_{i=1}^{N} l_i \; - \; n$. Combined with the N equations given by

$$\sum_{j=1}^{l_i} x_i^j = 1 \tag{1.61}$$

the equilibrium points of game $\Lambda^*(t)$, for $t > 0$, is given by the K^* equations of (1.60) and the N equations of (1.61). Let K^{**} be the total number of such equations then $K^{**} = K^* + N = \sum_{i=1}^{N} l_i$.

All equations which characterise the equilibrium points are algebraic equations in the variables s_i^j (each with probability x_i^j) and t. Define Z to be the set of all $(K^{**}+1)$-vectors $(t, \mathbf{p})$ satisfying the K^{**} equations of (1.61) and (1.60). Then let T be the subset of Z where all probabilities x_i^j also satisfy the additional requirement of $x_i^j \geq 0$. In particular T is the subset of Z which lies within the compact and convex polyhedron $R = P \times I$ where $I = [0, 1]$. T is called the *solution graph* for the set of equalities and inequalities it solves. For a given point $(t, \mathbf{p})$ the value t is the *first coordinate*. For game $\Lambda^*(t)$ the strategy space P is given by R^t: the set of all points in R with t as their first co-ordinate. The set E^t will be the set of all points in R^t which are equilibrium points of the game $\Lambda^*(t)$. Lastly T^t is the intersection of the solution graph T with R^t. Then by Lemma 1 for $0 < t \leq 1,\; E^t = T^t$ and in the original game when $t = 0$, $E^0 \subseteq T^0$.

Let Γ be a non-degenerate and quasi-strong game (which the paper proves accounts for almost all games) with finite number of players each with a finite pure strategy set. The set T is a solution graph and as such it contains a finite number of branches. The paper shows every equilibrium point $\mathbf{p}$ of Γ is an end point of some branch denoted by $\beta(\mathbf{p})$. Since there are two end points per branch the number of equilibrium points is finite.

Lemma 5 shows for a point $\mathbf{p} \in T$ there is a unique branch given by $\alpha(\mathbf{p})$ which originates from the point $(1, \mathbf{p'})$, where this point is shown to be the unique equilibrium point of $\Lambda^*(1)$. This branch is shown to terminate at the boundary of R with end point corresponding to $(0, \mathbf{p})$, i.e., an equilibrium point in Γ.

Any other equilibrium point in Γ is the end point of a branch which is shown to terminate with a second equilibrium point of Γ. Therefore the total number of equilibrium points in Γ is finite and odd.

1.8 Algorithms

Section 1.6 provides a broad range of applications of game theory but is far from being complete. One area yet to be discussed is the implications game theory has for computer science. In particular, the competition element within game theory makes it perfect for applications such as cryptography. One example in this area is the paper by Fischer and Wright [Fischer and Wright, 1993] who have described the analysis of a certain cryptographic system in terms of game theory. Game theory has also played its part in the internet, including e-commerce, and also within the wider scope of networks and routing problems. Game theory techniques can also be used to derive the lower bounds of algorithms with the players of the game being the algorithm developer and someone in an adversary role [Yao, 1977]. Additionally much work has been in done in developing algorithms to determine equilibrium points. The progress of this can be divided into two broad categories. The first being those algorithms which identify a single equilibrium point from a game, the algorithms resulting from the proofs in Section 1.7 belong to this category. The second group contains those algorithms which identify all equilibrium situations in the game and are called *Equilibrium Enumeration Methods* [Winkels, 1979]. In zero-sum games the problem of identifying an equilibrium point is equivalent to finding the solutions to a linear program [Dantzig, 1963]. However it has been shown by Savani and von Stengel [Savani and von Stengel, 2004] there is a class of bimatrix games such that the algorithm resulting from the Lemke and Howson paper [Lemke and Howson Jr, 1964] is always exponential. As the number of players in a game increases the corresponding algorithms become computationally harder. In fact Daskalakis, Goldberg and Papadimitriou showed in [Daskalakis et al., 2006] the problem of computing Nash equilibria in games with more than 4 players is PPAD-complete; where the PPAD class is a complexity class of search problems including those to find a Brouwer fixed point. It has since be shown by Chen and Den [Chen and Deng, 2006] that in fact the problem of finding a Nash Equilibria in all 2-player games is PPAD-complete. Since the problem of finding a Brouwer fixed point has exponential complexity, it must be that algorithms to identify Nash equilibria are too.

Chapter 2

Preliminary Definitions

In this Chapter the topological objects and Lemmas which will be fundamental to the original proofs in this Thesis are defined. We begin with the definition of a **simplex** as this will be one of the key building blocks in this Chapter and those which follow.

A 2-dimensional simplex is simply a triangle, a 1-dimensional simplex a straight line segment and a 0-dimensional simplex a single vertex. It is then natural to intuitively understand a simplex in n dimensions, or equivalently an n**-simplex**, to be the analogy of the triangle over $\mathbb{R}^n$. Before we can express this formally we need the following definition.

Definition 2.1 (Affinely Independence [Fulton, 1995])
A set of $n+1$ *points* $z_0, \ldots, z_n$ *in a vector space are* affinely independent *if there is no relation* $t_0z_0+t_1z_1+\cdots+t_nz_n = 0$ *with real numbers* $t_0, \ldots, t_n$ *satisfying* $t_0+t_1+\cdots+t_n = 0$ *with not all* $t_i = 0$.

Definition 2.2 (Geometric Simplex [Fulton, 1995])
For a set of $n+1$ *affinely independent points* $z_0, \ldots, z_n$ *the set*

$$\{t_0z_0 + t_1z_1 + \cdots + t_nPz_n \ \mid\ t_i \geq 0,\ t_0 + t_1 + \cdots + t_n \ = \ 1\} \tag{2.1}$$

is called the (geometric) closed simplex *of dimension* n *or* n-simplex

An open simplex *is defined by the set*

$$\{t_0 z - 0 + t_1 z_1 + \cdots + t_n z_n \ | \ t_i > 0, \ t_0 + t_1 + \cdots + t_n \ = \ 1\} \tag{2.2}$$

For $p \in \{0, \ldots, n\}$ a **p-face** of a n-simplex Δ is the p-simplex constructed using any $p+1$ of the $n+1$ affinely independent vertices $v_0, \ldots, v_n$ of a simplex Δ. If $p = n-1$ then the face is a **proper face** and the prefix 'p' is dropped. A 1-simplex, or **edge**, is the simplex using any 2 of the original points, and a 0-simplex, or **vertex**, is a single point.

Simplices can be 'connected together' to form a much larger topological object, a **simplicial complex**.

Definition 2.3 (Geometric Simplicial Complex)
A geometric simplicial complex *is the finite collection of (geometric) simplices such that any two simplices are either disjoint or share a common face. Additionally every face of a simplex in the simplicial complex must also belong to the simplicial complex.*

Definition 2.4 (Abstract Simplices and Simplicial Complex [Fulton, 1995])
A finite abstract simplicial complex *is a finite set V, called the vertices, and a collection K of subsets of V called the (abstract) simplices, with the property that every subset of a simplex is a simplex. Then K is the simplicial complex. An n-simplex is a set σ in K with $n+1$ elements.*

Observe the abstract simplicial complex is a combinatorial description of the geometric simplicial complex. The **dimension** of a simplicial complex is equal to the largest dimension of any simplex it contains and is said to be open or closed if the simplices it contains are open or closed. A simplicial complex is **labelled** if each vertex is assigned a unique label from a finite set of possibilities.

Definition 2.5 (Sub-complex [Fulton, 1995])
A sub-complex *L of a simplicial complex K is a subset of the simplices in K such that whenever a simplex Δ is in L, so are all its faces; then L is a simplicial complex with vertex set being a subset of the vertices of K.*

Definition 2.6 (Boundary)
If Δ is a closed simplex let Δ^0 denote its interior. Then the boundary *of Δ is*

$$\partial\Delta = \Delta \setminus \Delta^0. \tag{2.3}$$

For an open simplex Δ let $\bar{\Delta}$ denote its closure. In this case

$$\partial\Delta = \bar{\Delta} \setminus \Delta. \tag{2.4}$$

Then the boundary of a simplicial complex of dimension n can be identified as the union of the faces of dimension $(n-1)$ which belong to just one maximal simplex.

The definition of a simplicial complex can be tightened to satisfy some additional properties.

Definition 2.7 (Pseudomanifold [Spanier, 1966])
An n-dimensional pseudomanifold is a simplicial complex K such that

1. *Every simplex in K is the face of some n-simplex in the complex. (Note that K itself is contained in this definition)*
2. *Every $(n-1)$-simplex of K is the face of at most two n-simplices of K.*
   This is the property of **non-ramification**.
3. *If s and s' are n-simplices of K, then there is a finite sequence $s = s_1, \ldots, s_l = s'$ of n-simplices of K such that s_i and s_{i+1} have an $(n-1)$-face in common for $1 \leq i < (l-1)$.*

The boundary of an n-dimensional pseudomanifold K is defined to be the sub-complex of K generated by the set of $(n-1)$-simplices which are faces of exactly one n-simplex of K.

To enable us to connect multiple simplicial complexes together we introduce the **join** operator which relies on the following definition.

Definition 2.8 (Disjoint Union [Spanier, 1966])
If $J = \{j\}$ is a set and $\{A_j\}$ is a family of sets indexed by J, their disjoint union, *is denoted by*

$$\bigvee_{j \in J} A_j \quad := \quad \bigcup_{j \in J} (j \times A_j) \tag{2.5}$$

Equivalently the disjoint union of two sets combines the distinct elements of each set while ensuring the original set membership is distinguishable in the union set.

Definition 2.9 (Join of Simplicial Complexes [Spanier, 1966])
If K_1 and K_2 are simplicial complexes, their join $K_1 * K_2$ *is the simplicial complex defined by*

$$K_1 * K_2 \quad := \quad K_1 \vee K_2 \;\cup\; \{s_1 \vee s_2 \,|\, s_1 \in K_1,\, s_2 \in K_2\} \tag{2.6}$$

Thus the set of vertices of $K_1 * K_2$ is the disjoint union of the vertex sets of K_1 and K_2.

This definition will be illustrated in Example 3.2.

We now state, and provide our proof of, a Lemma defining the boundary of the join of two simplicial complexes.

Lemma 2.10 (Boundary of Join)
Let $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}$ be two pseudomanifolds with $\dim(\mathcal{C}^{(i)}) = d_i$*. Then $\partial(\mathcal{C}^{(1)} * \mathcal{C}^{(2)})$ is generated by all $(d_1 + d_2)$-dimensional simplices of the kind $\Sigma^{(1)} * \Sigma^{(2)}$, where $\Sigma^{(i)}$ lies in $\mathcal{C}^{(i)}$ for every $i \in \{1, 2\}$ and $\Sigma^{(i)}$ lies in $\partial\mathcal{C}^{(i)}$ for one $i \in \{1, 2\}$.*

Proof. By definition the dimension of $(\mathcal{C}^{(1)} * \mathcal{C}^{(2)})$ is equal to $d_1 + d_2 + 1$. Therefore the dimension of the boundary $\partial(\mathcal{C}^{(1)} * \mathcal{C}^{(2)})$, must equal $d_1 + d_2$. Let Σ lie in $\mathcal{C}^{(1)} * \mathcal{C}^{(2)}$ such that $\dim(\Sigma) = d_1 + d_2$ and represent Σ as the join of two simplices $\Sigma^{(1)} * \Sigma^{(2)}$, where $\Sigma^{(i)}$ lies in $\mathcal{C}^{(i)}$ for every $i \in \{1, 2\}$. We show Σ lies in $\partial(\mathcal{C}^{(1)} * \mathcal{C}^{(2)})$ if and only if one of $\Sigma^{(i)}$ lies in $\partial\mathcal{C}^{(i)}$.

The conditions of dimension give

$$\dim \Sigma \quad = \quad \dim \Sigma^{(1)} + \dim \Sigma^{(2)} + 1 \quad = \quad d_1 + d_2. \tag{2.7}$$

It therefore follows that either $\dim \Sigma^{(1)} = d_1 - 1$ while $\dim \Sigma^{(2)} = d_2$ or $\dim \Sigma^{(1)} = d_1$ and $\dim \Sigma^{(2)} = d_2 - 1$. Consider for definiteness the first possibility. If $\Sigma^{(1)}$ does not lie in the boundary then there are two d_1-dimensional simplices, say Δ' and Δ'', in $\mathcal{C}^{(1)}$ such that $\Sigma^{(1)}$ is their common face. This is equivalent to Σ being the common face of simplices $\Delta' * \Sigma^{(2)}$ and $\Delta'' * \Sigma^{(2)}$, and therefore to Σ not belonging to $\partial(\mathcal{C}^{(1)} * \mathcal{C}^{(2)})$. This is also true when considering the second possibility. Therefore Σ lies in $\partial(\mathcal{C}^{(1)} * \mathcal{C}^{(2)})$ exactly when for one $i \in \{1, 2\}$, $\Sigma^{(i)}$ belongs to $\partial\mathcal{C}^{(i)}$

□

Definition 2.11 (Nerve [Fulton, 1995])
If $\mathcal{U} = \{U_v | \, v \in V\}$ is a finite collection of open sets whose union is a space X, define a simplicial complex, called the nerve *of $\mathcal{U}$ by taking V to be the vertices and defining the simplices to be the subsets S such that the intersection of the U_v for v in S is nonempty.*

Definition 2.12 (Star [Fulton, 1995])
If K is any simplicial complex and v is a vertex in K, define an open set $St(v)$ in K, called the star of v *to be the union of "interiors" or the simplices that contain v i.e., $St(v)$ is the complement in K of the union of those simplices Σ for which the vertex set of Σ does not contain v.*

We now provide our definition of a '*non-degenerate*' simplicial complex.

Definition 2.13 (Non-Degenerate Simplicial Complexes)
A simplicial complex K is said to be non-degenerate *if it satisfies the conditions of non-ramification.*

Recall, from Definition 2.11, the construction of a nerve (a simplicial complex) relies upon some finite covering of a space X. It is then natural to expect the properties of this covering to affect the degeneracy status of the nerve. Therefore our definition of a **non-degenerate nerve** will take account of this and as such will require the following definition and operations.

Definition 2.14 (Simple Subdivision)
Let Δ be a d-dimensional simplex with vertex set $\{v_1, \ldots, v_{d+1}\}$ and let w be its barycenter. Then the simple subdivision *of Δ is a simplicial complex consisting of simplices of the kind $\{w, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+1}\}$, for all $i = \{1, \ldots, d+1\}$, and all faces of Δ.*

Example 2.15
Figure 2.1 shows the simple subdivision of a 2-dimensional simplex

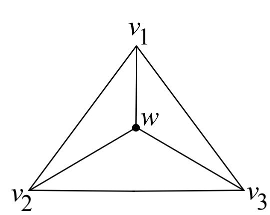


Figure 2.1: Simple subdivision of a 2-dimensional simplex

(End Example)

Definition 2.16 (Detachment Operation)
Let the $(d-1)$-dimensional simplex Λ with vertex set $\{v_1, \ldots, v_d\}$ be the common face of exactly k d-dimensional simplices $\Delta_1, \ldots, \Delta_k$ in a simplicial complex K. The result of the detachment operation *in K, of $\Delta_1, \ldots, \Delta_k$ with respect to Λ is a simplicial complex K' obtained from K in the following way. Suppose $w_1, \ldots, w_k$ are barycenters of $\Delta_1, \ldots, \Delta_k$ respectively. Refine K by taking simple subdivisions of $\Delta_1, \ldots, \Delta_k$, removing simplices $\{w_i, \Lambda\}$, $i = 1, \ldots, k$ from K and identifying pairs of simplices*

$$\{w_1, v_1, \ldots, v_{l-1}, v_{l+1}, \ldots, v_d\}, \ldots \{w_k, v_1, \ldots, v_{l-1}, v_{l+1}, \ldots, v_d\}$$

for all $l = \{1, \ldots, d\}$. In particular, vertices w_i, w_j become identical. In the case the complex K is labelled, the new vertex $w_i = w_j$ is assigned any label among the labels of the vertices $v_1, \ldots, v_d$.

Definition 2.17 (Regularisation)
The process of regularisation *of a d-dimensional simplicial complex K is the removal of all maximal simplices from K' which, after finite applications of the detachment operation, no longer share any $d-1$-face with any other simplex in K'.*

Remark
In a simplicial complex K consider two simplices Δ_i, Δ_j where $i \neq j$ and let $\{\Delta_1, \ldots, \Delta_{k'}\}$ be the set of simplices (not containing Δ_j) which have a non-empty intersection with Δ_i. Then observe if the detachment (or regularisation) operation is applied to Δ_i and Δ_j then every intersection between Δ_i and any simplex in $\{\Delta_1, \ldots, \Delta_{k'}\}$ is maintained.

Example 2.18
Figure 2.2 shows a simplicial complex failing the conditions of non-ramification. It is clear all three simplices share the face $\{v_1, v_3\}$.

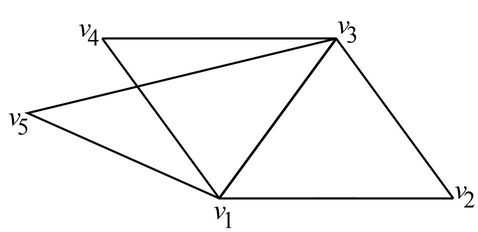


Figure 2.2: Simplicial complex to be regularised

The regularisation of the simplicial complex in Figure 2.2 begins with detachment operation and the simple subdivision of simplices $\{v_1, v_3, v_4\}$ and $\{v_1, v_3, v_5\}$ with barycenters of w_1 and w_2 respectively. This is shown in Figure 2.3.

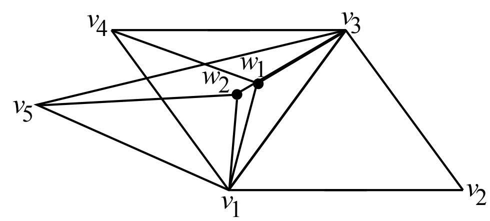


Figure 2.3: Regularisation: step 1

From the simplicial complex in Figure 2.3 remove simplices $\{w_1, v_1, v_3\}$ and $\{w_2, v_1, v_3\}$. Let $w_1 = w_2$ and pair simplices $\{w_1, v_1\}$ with $\{w_2, v_1\}$ and $\{w_1, v_3\}$ with $\{w_2, v_3\}$. Then the face $\{v_1, v_3\}$, from simplex $\{v_1, v_2, v_3\}$, no longer belongs to a second maximal simplex. Further, in this example, since faces $\{v_1, v_2\}$ and $\{v_2, v_3\}$ also do not belong to a second simplex, simplex $\{v_1, v_2, v_3\}$ is detached from the simplicial complex and thus is removed. The resulting simplicial complex is then regularised and can be seen in Figure 2.4.

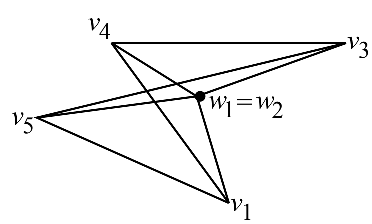


Figure 2.4: Regularisation: step 2

(End Example)

Definition 2.19 (Non-Degenerate Nerves)
A nerve is said to be non-degenerate *if after regularisation the resulting simplicial complex satisfies the conditions of non-ramification.*

Remark
Regularisation of a nerve has the affect of perturbing the intersections of the sets forming the underlying covering.

Definition 2.20 (Regular)
A nerve is regular *if it satisfies the conditions of non-ramification, and in particular is non-degenerate, without the need for regularisation.*

For the remainder of this Chapter we move away from topological definitions and concentrate on a Lemma which will be key to our work.

Lemma 2.21 (Sperner's Lemma)
For a simplex Δ of dimension $(n-1)$ label each vertex uniquely from the set $\{0,\ldots,n-1\}$ and form a finite non-ramified simplicial complex within Δ. If an unlabelled vertex is identified to lie in a face $\Lambda \subset \Delta$, where $1 \leq \dim \Lambda \leq (n-1)$, label the vertex by any of the labels assigned to the vertices of Λ. Then in the resulting simplicial complex there are an odd number of $(n-1)$-simplices with vertices labelled by $\{0,\ldots,n-1\}$, also called **completely labelled simplices**.

Proof. The proof of Sperner's Lemma is by induction. Let Δ_i be an i-dimensional simplex with vertices labelled by $\{0,\ldots,i\}$. A simple observation confirms any simplicial complex contained within Δ_1 contains an odd number of completely labelled simplices. For the induction hypothesis we assume the result is true for Δ_{n-1}. The simplex Δ_{n-1} is a face of Δ_n and in particular each completely labelled simplex from Δ_{n-1} is contained in the boundary of Δ_n. Then, by the property of non-ramification, the completely labelled simplices of Δ_{n-1} must be the face of exactly 1 n-simplex in Δ_n. Denote such simplices by Σ_j for $j \in J$ where the size of indexing set J is equal to the number of completely labelled simplices in Δ_{n-1} and therefore, by assumption, $|J|$ is odd. Observe all faces on the boundary of Δ_n labelled by $\{0,\ldots,n-2\}$ must be constructed in this way. Two simplices in Δ_n are said to be adjacent if they share a face labelled by $\{0,\ldots,n-2\}$. Select simplex Σ_1 and construct its path of adjacent simplices. If Σ_1 is completely labelled there is no adjacent simplex, so assume this is not the case. Then by the property of non-ramification any simplex can have at most two faces with vertex set $\{0,\ldots,n-2\}$ and since, by definition, Σ_1 has one such face lying in the boundary the first choice of adjacent simplex must be unique. Let the adjacent simplex be given by Σ_1^1. Then either Σ_1^1 is completely labelled and the path terminates or there is another face labelled by $\{0,\ldots,n-2\}$ which is not shared by Σ_1. If this is the case then move to the next adjacent simplex given by Σ_1^2. The path will either terminate with Σ_j for $j \in J, \quad j \neq 1$ or with a completely labelled simplex. Since there are an odd number of simplices Σ_j, an odd number of paths must end with a completely labelled simplex. However these paths may not identify all completely labelled simplices in Δ_n. If this is the case take such a simplex as a start point for a path then it is clear to see this path must terminate with a second completely labelled simplex. Thus the total number of completely labelled simplices in Δ_n is odd. $\square$

The following Lemma is an abstract form of Sperner's Lemma and is given as written in [Vorob'ev, 1994][1]. The proof which follows is also from the same publication.

Lemma 2.22
Consider any finite un-orientated graph $G = (V, E)$ with set V of vertices and set E of edges where each vertex $v \in V$ has $\deg(v) \leq 2$. From the set V we select a subset $\mathcal{N} \subset V$ of vertices which we call **normal***, where if $v \in \mathcal{N}$ then $\deg(v) \in \{0, 1\}$. A vertex $v \in V \setminus \mathcal{N}$ is* **extreme** *if $\deg(v) = 1$ while a vertex $v \in \mathcal{N}$ is* **extreme** *if $\deg(v) = 0$. Then the number of normal vertices in a graph G has the same parity as the number of extreme vertices. Figure 2.5 provides a graphical representation of this Lemma.*

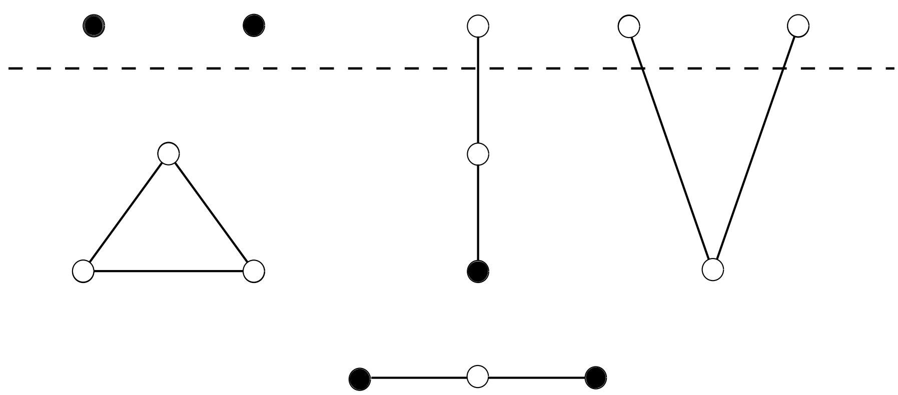


Figure 2.5: Representation of Lemma 2.22, here the normal vertices are shown in black and the extreme vertices are above the dashed line, taken from [Vorob'ev, 1994]

Remark
The **degree** of a vertex v, $\deg v$, is equal to the number of edges incident to it.

Proof. Assume G has p vertices so the vertex set V is given by $\{v_j \,|\, j = 1, \ldots, p\}$. To each vertex $v_j \in V$ assign the value $a_j = \{1, 2\}$ according to the rules

$$a_j = \begin{cases} 1 & \text{if } v_j \in \mathcal{N} \\ 2 & \text{if } v_j \in V \setminus \mathcal{N} \end{cases} \tag{2.8}$$

[1]Section 3.16, Chapter 2

Set $a = \sum_{j=1}^{p} a_j$. Then if the total number of normal vertices is odd a must be odd too, similarly when the number of normal vertices is even. From (2.8), if v_j is normal then $a_j = 1$. Observe if v_j is not extreme, then the degree of this vertex is 1 and we have $a_j = \deg(v_j)$. If on the other hand v_j is extreme, the degree of the vertex is 0 and $a_j = \deg(v_j) + 1$. Identical results are achieved by analysing the case when v_j does not belong to the set $\mathcal{N}$. In particular

$$a_j \;\; = \;\; \begin{cases} \deg(v_j) & \text{if } v_j \text{ is not extreme} \\ \deg(v_j) + 1 & \text{if } v_j \text{ is extreme} \end{cases} \tag{2.9}$$

Therefore if G contains q extreme vertices

$$a \;\; = \;\; \sum_{j=1}^{p} a_j \;\; = \;\; \sum_{j=1}^{p} \deg(v_j) + q \tag{2.10}$$

Now invoke a popular result from graph theory which states the sum of the degrees of all vertices is even. Therefore the parity of a is equal to the parity of q. However it has already been shown the parity of a is equal to the parity of normal vertices. Consequently the number of normal vertices in a graph G has the same parity as the number of extreme vertices.

$\square$

In Lemma 2.22 replace the extreme vertices of the graph G with the simplices Σ_j as defined in the proof of Lemma 2.21. Then replace the notion of normal vertices with the definition of a completely labelled simplex of Δ. It is then clear how Lemma 2.22 is an abstract form of Lemma 2.21

Chapter 3

Simplicial Complexes and Nash's Theorem

3.1 Motivation and Objectives

An important concept within game theory is the definition of a solution; that is an answer to the question 'Which strategy should each player select in order to achieve the most optimal position within the game?'. Of course interpretation of this, and consequently the resulting definition of a solution, can be (and has been) subjective. However there is wide acceptance that those situations which are said to be *equilibrium* do in fact solve the game in the required manner. As a result of the different variants of the game theory model, the number of definitions of a equilibrium situation is numerous. In a non-cooperative game the solutions are those points satisfying the definition of Nash equilibria, as given in Definition 1.10, and when the payoff functions of a game are polylinear, Nash's Theorem, Theorem 1.13 ensures such equilibrium points exists. This is clearly an invaluable tool when analysing a non-cooperative game and will provide the focus for this Thesis.

The proofs outlined in Section 1.7 verify Nash's Theorem for non-cooperative games but all require some geometric considerations and require the polylinear property of the payoff function. We investigate these two things. In particular we aim to produce a combinatoric proof of Nash's Theorem and show Nash's Theorem still holds when the payoff functions are not expectations.

The set of all payoff functions determine each player's outcome for a given situation of a game. Furthermore, in an N-player game, if the strategy choices for all player's $\{1, \ldots, i-1, i+1, \ldots, N\}$ are known then this information can be used to determine which strategies will produce the best repsponse for player i. It is this property that allows the identification of equilibrium points which, by definition, are situations where no player achieve an improved outcome by altering their strategy choice only. It is clear polylinear functions are not the only functions which, for a given input, allow identification of an optimal outcome. Therefore given the set of equilibrium situations are the solutions to a finite set of inequalities it seems unlikely Nash's Theorem is a special consequence of the property of polylinearity. Subsequently there must be a less restrictive constraint on the payoff functions which will still permit the determination of equilibrium situations and satisfy Nash's Theorem.

Remark
Those games with payoff functions which are not polylinear will be abstract in nature.

At an equilibrium situation each player's strategy is optimal with respect to the strategies chosen by the other players. For an N-player game Γ, and for all $i = \{1, \ldots, N\}$, let $p_i \in P_i$ be a mixed strategy for player i with support Y_i. Then the situation $(p_1, \ldots, p_N)$ is an equilibrium situation if and only if for all players i the set of pure best responses over $(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N)$ is a superset of Y_i. This can be determined by purely knowing which strategy is favoured at a given point, and the actual payoff value is not required. In light of this, and our discussion in the previous paragraph, in this Thesis we examine a new category of game where for all $i = \{1, \ldots, N\}$ each payoff function G_i is replaced by a **total order** (defined with restrictions) denoted by $\succ_i$. Observe, for each player i, total orders will continue to allow player i to identify his optimal strategies from P_i for all situations from $P^{(i)}$. This leads us to establish our first objective for this Thesis:

Objective 1
To prove Nash's Theorem for N-player non-cooperative games Γ where each polylinear payoff function G_i is replaced by a total order $\succ_i$ for all $i = \{1, \ldots, N\}$.

The notion of a total order will be too general for games where an actual payoff value is required. However proving Nash's Theorem for a total order over the set of mixed strategy situations will verify the result for any payoff function from which a total order can be ascertained. This will provide a substantial increase in the number and range of 'games' for which Nash's Theorem can be applied. In achieving this objective we will have provided a new proof of Nash's Theorem as given in Theorem 1.13.

The representation of all payoff functions G_i automatically provide a geometric setting in which the outcomes, and thus solutions, of the game can be identified. Clearly this will no longer exist when the payoff functions are replaced with total orders. In particular the use of a total order instead of a payoff function will provide an abstract definition of a game. However we will be able to use this new representation to aid our insight into the foundations of Nash's Theorem and to address why this result is true.

On the successful proof of Nash's Theorem for a total order we will have demonstrated the result is not a consequence of the properties of a payoff function. Given this has been a significant focus for all other proofs of Nash's Theorem, this will be an important step forward in establishing its origins. However this will also lead to a much larger question; if the proof of Nash's Theorem is not founded on the properties of the payoff functions of a game, then what are the underlying mathematical reasons for this phenomena? Equivalently what special properties of game theory allow Nash's Theorem to be true?

For all $i = \{1, \dots, N\}$, a simple observation allows us to see each total order $\succ_i$ (and consequently the payoff function G_i) can be translated into a covering of $P^{(i)}$. This allows the construction of a simplicial complex representation of a game where vertices and edges represent the adjacency conditions of the covering elements. This is now a very different form to the traditional representation of a game and leads to the question is there a more general mathematical model for which an equivalent version of Nash's Theorem exists, and, for which game theory is a specific example? If so it would appear to be a simplicial complex. Therefore our second objective will be:

Objective 2
To identify the origins of Nash's Theorem by defining a simplicial complex $\mathcal{S}$ such that:

1. A more general form of Nash's Theorem can be proved for $\mathcal{S}$.

2. The traditional statement of Nash's Theorem for games Γ can be attained from $\mathcal{S}$.

3.2 Plan of Work

Our original work in this Thesis will begin with the definition of a simplicial complex $\mathcal{S}$. This simplicial complex will be defined independently from any notion of game theory and will be shown to contain a finite and odd number of, what we shall define to be, **equilibrium simplices**. Our return to game theory is marked with our description of a simplicial complex representation of an N-player non-degenerate, non-cooperative game, where all equilibrium situations from Γ correspond exactly to the set of equilibrium simplices. In relation to this we discuss why replacing the payoff function G_i with a total order $\succ_i$ does not alter this construction procedure in any way. Those games which rely on the total order $\succ_i$ will be referred to as **generalised games**. Finally by demonstrating our simplicial complex representation of a game satisfies the definition of $\mathcal{S}$, Nash's Theorem (for generalised games) will be proved. The use of simplicial complexes allows us to produce a purely combinatoric proof of Nash's Theorem for non-degenerate games.

We then provide an alternative definition of the simplicial complex $\mathcal{S}$ which can be used to prove Nash's Theorem for an important subset of non-cooperative games, bimatrix games. This new definition will be simpler and more intuitive than $\mathcal{S}$ and will highlight the problem of increasing complexity in representing games when the number of players exceeds 2. We explain why our new definition does not extend naturally to the N-player case and demonstrate how it is a particular example of the original simplicial complex $\mathcal{S}$. To conclude our work we provide some examples of generalised games. For a clear picture of how our work is a deviation and improvement to previous proofs we follow this with a discussion comparing our work to the proofs given in Section 1.7. Finally we provide some thoughts on how our work can be developed and taken advantage of in future research.

Throughout this Thesis we will comment on the similarities between our work and the Lemke-Howson constructive procedure presented by Shapley [Shapley, 1974]. In particular we hope this will clearly enable the reader to appreciate how the work in this Thesis is a generalisation of the Lemke-Howson constructive procedure [Lemke and Howson Jr, 1964].

In this Chapter we define the non-degenerate simplicial game complex $\mathcal{S}$ and provide our proof that such (non-degenerate) complexes contain an odd number of equilibrium simplices.

3.3 Defining the Complex

We begin by providing the notations we will use to define our non-degenerate simplicial game complex $\mathcal{S}$. For a finite natural number m, our simplicial game complex $\mathcal{S}$ will rely on the partitioning of the set $\mathbf{m} := \{1, \ldots, m\}$ into n subsets. For $i = \{1, \ldots, n\}$ each partition element is defined by $\mathbf{m_i} := \{m_{i-1} + 1, \ldots, m_i\}$ with $m_0 = 0$, $m_n = m$ and $|\mathbf{m_i}| > 1$ such that $\mathbf{m} = \mathbf{m_1} \cup \cdots \cup \mathbf{m_n}$ and all sets are pairwise disjoint. Let V denote the set of vertices in $\mathcal{S}$ and be subject to the condition $|V| \geq m + n$. We define $\phi : V \mapsto \mathbf{m}$ to be a surjective **labelling function** which assigns each vertex in $\mathcal{S}$ a unique label from a set of m possibilities. We are then able to identify and fix a subset $V' \subset V$ such that the restriction of ϕ to V' is bijective. Within the set $V \setminus V'$, and for every i, we insist on being able to identify a non-empty subset of vertices W_i such that $\phi(W_i) \subset \mathbf{m_i}$.

Notation
For a simplex Σ let $v(\Sigma)$ denote its vertex set.

Let Σ_{m_i} be the simplex with vertices in V' such that $\phi(v(\Sigma_{m_i})) = \mathbf{m_i}$ then

$$\dim \Sigma_{m_i} \;=\; |\mathbf{m_i}| - 1 \;=\; (m_i - m_{i-1} - 1) \tag{3.1}$$

Let $\bar{\Sigma}_{m_i}$ be the closure of Σ, i.e. the complex consisting of Σ_{m_i} and all its faces, and let $\partial\Sigma_{m_i}$ be the boundary of Σ_{m_i} i.e., $\bar{\Sigma}_{m_i} \setminus \Sigma_{m_i}$.

The finite (combinatorial) simplicial game complex $\mathcal{S}$ with set of vertices V will be defined by induction on the number of partition element used in its construction. Therefore for $k = \{1, \ldots, n\}$ let $V_k \subset V$ satisfy $|V_k| \geq \left(\sum_{j=1}^{k}(m_j - m_{j-1}) + k\right) = m_k + k$ such that V_k contains all elements from V which are mapped to the set $\mathbf{m_1} \cup \cdots \cup \mathbf{m_k}$ under ϕ. Then $\phi_k : V_k \mapsto \mathbf{m_1} \cup \cdots \cup \mathbf{m_k}$ is the surjective labelling function ϕ restricted to input set V_k. Fix $V_k' \subset V_k$ such that the restriction $\phi_k|_{V_k'}$ is bijective. Finally let $W_k = V_k \setminus V_k' \neq \emptyset$ where $\phi_k(W_k) = \phi(W_k) \subset \mathbf{m_k}$, then this coincides with its previous definition.

For $k \in \{1, \ldots, n\}$ we now use these preliminary notations to define the finite (combinatorial) simplicial game complex $\mathcal{S}_k$ with set of vertices V_k.

Remark
When $k = n$ the original notations are recovered. In particular $\mathcal{S} = \mathcal{S}_n$.

Definition 3.1 (Non-Degenerate Simplicial Game Complex $\mathcal{S}_k$)
Define a non-degenerate simplicial game complex $\mathcal{S}_k$ of the order k *as an* $(m_k - 1)$-*dimensional simplicial complex with finite set of vertices* $V_k = V_k' \cup \bigcup_{j=1}^{k} W_j$ *such that*

1. *The subcomplex with vertices* V_k' *coincides with* $\partial\Sigma_{m_1} * \cdots * \partial\Sigma_{m_k}$.
2. *For all* $j = \{1, \ldots, k\}$ *each maximal simplex in* $\mathcal{S}_k$ *contains at least one vertex from the set* W_j.
3. *For all* $j = \{1, \ldots, k-1\}$ *any vertex with label from* $\mathbf{m_j}$ *must belong to* $\mathcal{S}_{k-1}$.
4. $\mathcal{S}_k$ *is an* $\left(\sum_{j=1}^{k}(m_j - m_{j-1}) - 1\right) = (m_k - 1)$-*dimensional* simplicial pseudomanifold with boundary *where the boundary is defined as follows.*

Let Δ_k *be a maximal (i.e.,* $(m_k - 1)$*-dimensional) simplex in* $\mathcal{S}_k$*, and for every* $j = \{1, \ldots, k\}$ *let* $W_{\Delta_k, j}$ *be the set of all vertices from* Δ_k *which belong to* W_j*. Then by definition* $W_{\Delta_k, j} \neq \emptyset$ *for all* $j \in \{1, \ldots, k\}$*. Let* $\Sigma_{\Delta_k, j}$ *be the (possibly empty) set of vertices from* Δ_k *which belong to* Σ_{m_j}*. Then an* $(m_k - 2)$*-dimensional face* Λ_k *in* $\mathcal{S}_k$ *belongs to the boundary of* $\mathcal{S}_k$ *iff there exists a maximal simplex* Δ_k *in* $\mathcal{S}_k$ *and an* $j \in \{1, \ldots, k\}$ *such that* Λ_k *is a face of* Δ_k *obtained by removing a vertex* x*, with* $\{x\} = W_{\Delta_k, j}$*, from* $v(\Delta_k)$*. This boundary definition is illustrated in Example 3.2*

Remark
For the subsequent considerations the condition to be a pseudomanifold with boundary can be replaced by a weaker condition to be a *non-ramified complex with boundary* i.e. satisfying axiom 2 of Definition 2.7.

Remark
In line with Definition 2.13, if $\mathcal{S}$ is a not a non-ramified complex then $\mathcal{S}$ is degenerate and fails property (4) of Definition 3.1.

Example 3.2 (Illustrating The Boundary)
We illustrate the boundary of a simplicial game complex of the order 2, $\mathcal{S}_2$. Assume $\mathbf{m} = \{1, 2, 3, 4\}$ is partitioned into the two sets, $\mathbf{m_1} = \{1, 2\}$ and $\mathbf{m_2} = \{3, 4\}$. Then the simplices Σ_{m_1} and Σ_{m_2} are both 1-dimensional with vertices labelled by $\mathbf{m_1}$ and $\mathbf{m_2}$ respectively. By definition $\mathcal{S}_2$ must contain the join of the boundaries of these simplices (which in this case will be the vertices) and the result of this can be seen in Figure 3.1

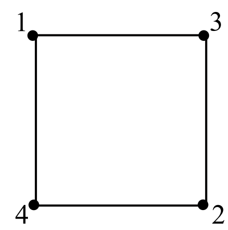


Figure 3.1: $\partial\Sigma_{m_1} * \partial\Sigma_{m_2}$

For ease of representation assume the simplicial complex $\mathcal{S}_2$ can be represented as the join of two smaller simplicial complexes T_1 and T_2 where in this case $\dim T_1 = \dim T_2 = 1$ (we prove this is possible for a simplicial complex of the order 2 in Chapter 5). We assume the simplicial complexes T_1 and T_2 are as given in Figure 3.2. Those vertices which belong to just one edge are the boundary vertices of the simplicial complexes T_i.

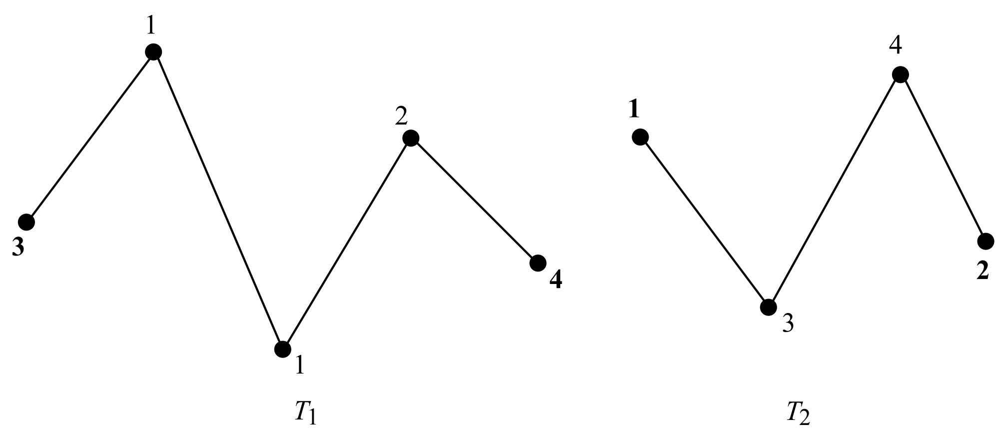


Figure 3.2: The simplicial complexes T_1 and T_2

Then a maximal simplex in $\mathcal{S}_2$ is the result of a maximal simplex from T_1 joined to a maximal simplex from T_2. The table in Figure 3.3 describes the vertex set of all maximal simplices in $\mathcal{S}_2$. Column T_1 describes the simplex from T_1 as shown in Figure 3.2, similarly for T_2. For $i = \{1, 2\}$ the vertex set of each simplex Δ in $\mathcal{S}_2$ is divided into sets $W_{\Delta,i}$ and $\Sigma_{\Delta,i}$ as defined in Definition 3.1. In Figure 3.3 the columns marked $W_{\Delta,i}$ contain the vertices which belong to $W_{\Delta,i}$ and the column marked Σ contains those vertices which belong to $\Sigma_{\Delta,1} \cup \Sigma_{\Delta,2}$. The final column details the faces of $\mathcal{S}_2$ which Definition 3.1 depict should be boundary faces. The vertex labels given in bold in the first two columns and last column highlight their position as a boundary vertex.

Using Lemma 2.10 we can identify which simplices in $\mathcal{S}_2$ should have a face on the boundary. Recall this Lemma tells us a boundary face Λ will be the join of two simplices $C^{(1)}$ and

T_1	T_2	$W_{\Delta,1}$	$W_{\Delta,2}$	Σ	Boundary Faces
3,1	**1**,3	1	3	1,3	{**1**,**3**,1}, {**1**,**3**,3}
3,1	3,4	1	3,4	3	$\{\mathbf{3}, 3, 4\}$
3,1	4,**2**	1	4	2,3	$\{\mathbf{2}, \mathbf{3}, 1\}, \{\mathbf{2}, \mathbf{3}, 4\}$
1,1	**1**,3	1,1	3	**1**	$\{\mathbf{1}, 1, 1\}$
1,1	3,4,	1,1	3,4	$\emptyset$	-
1,1	4,**2**	1,1	4	2	$\{\mathbf{2}, 1, 1\}$
1,2	**1**,3	1,2	3	1	$\{\mathbf{1}, 1, 2\}$
1,2	3,4	1,2	3,4	$\emptyset$	-
1,2	4,**2**	1,2	4	2	$\{\mathbf{2}, 1, 2\}$
2,**4**	**1**,3	2	3	1,4	$\{\mathbf{1}, \mathbf{4}, 2\}, \{\mathbf{1}, \mathbf{4}, 3\}$
2,**4**	3,4	2	3,4	4	$\{\mathbf{4}, 3, 4\}$
2,**4**	4,**2**	2	4	2,4	$\{\mathbf{2}, \mathbf{4}, 2\}, \{\mathbf{2}, \mathbf{4}, 4\}$

Figure 3.3: Table describing the vertex set of all maximal simplices in $\mathcal{S}$

$C^{(2)}$ such that $C^{(i)}$ belongs to the simplicial complex T_i and $C^{(j)}$ to the boundary T_j for $i, j \in \{1, 2\}$, $i \neq j$. That is either $C^{(1)}$ is a simplex of T_1 and $C^{(2)}$ is a simplex of ∂T_2 or $C^{(1)}$ is a simplex of T_2 and $C^{(2)}$ is a simplex of ∂T_1.

In our example a simplex from ∂T_i is a single vertex. Therefore, following Lemma 2.10 if a simplex from $\mathcal{S}_2$ lies in the boundary we will be able to identify a face, given by three vertices, such that one belongs to ∂T_i and the other one to T_j for $i \neq j$. Observe Figure 3.3 shows this coincides with our expectations of the boundary taken from Definition 3.1. Additionally if a simplex from $\mathcal{S}_2$ contains one boundary vertex, then since at least one vertex must come from both of W_1 and W_2, Lemma 2.10 tells us to expect to be able to identify exactly one boundary face. Similarly if a simplex contains two boundary vertices then we can expect two boundary faces. Observe in this example this is the maximum number of boundary vertices we can find in any one simplex. This analysis ties in with the boundary faces of $\mathcal{S}_2$ as given in Figure 3.3.

(End Example)

In order to prove an analogy of Nash's Theorem for the simplicial game complex $\mathcal{S}_n$ we define **equilibrium simplices**

Definition 3.3 (Equilibrium Simplex)
A simplex Σ in $\mathcal{S}_k$ of dimension $(m_k - 1)$ is said to be equilibrium *if $\phi(v(\Sigma)) = \{1, \ldots, m_k\}$ and* sub-equilibrium *if $\phi(v(\Sigma)) = \{1, \ldots, m_k - 1\}$.*

Definition 3.4 (Sub-Equilibrium Face)
A face of Σ *with vertex set* $\{1, \ldots, m_k - 1\}$ *is called* sub-equilibrium. *Observe such faces will only belong to equilibrium or sub-equilibrium simplices.*

This completes the definition of our simplicial complex $\mathcal{S}_k$. We now formulate Nash's Theorem for simplicial game complexes.

Theorem 3.5 (Nash's Theorem for Simplicial Game Complexes)
Every simplicial game complex $\mathcal{S}$ *contains an equilibrium simplex. When* $\mathcal{S}$ *is non-degenerate then the number of such simplices is finite and odd.*

Remark
We make our first comparison to the Shapley paper [Shapley, 1974], discussed in Chapter 1.7. In particular we recall the nodes representing equilibrium situations are required to be completely labelled. In our simplicial game complex we define our object of interest as completely labelled simplices.

3.4 Preliminary Results

Before we prove Theorem 3.5 for the non-degenerate case, we provide some important preliminary definitions and results.

Definition 3.6 (Subcomplex $\mathcal{S}'$)
For $k = \{2, \ldots, n\}$ *let* $\mathcal{S}'$ *be a subcomplex of* $\mathcal{S}_k$ *constructed as follows. Consider the face* σ_k *of* Σ_{m_k} *such that* $\phi_k(v(\sigma_k)) = \{m_{k-1}+1, \ldots, m_k - 1\}$ *and construct the closure of the star of* σ_k *denoted by* $\overline{St}(\sigma_k)$. *From* $\overline{St}(\sigma_k)$ *remove the (open) star of* σ_k *and the (open) stars of all vertices belonging to* W_k, *then* $\mathcal{S}'$ *is the remaining simplicial complex from* $\overline{St}(\sigma_k)$.

We illustrate this definition with the following example.

Example 3.7

Consider the nerve (recall Definition 2.11, also described in more detail in Chapter 4) of the covering in Figure 3.4 (this is actually the covering of a dyadic bimatrix game).

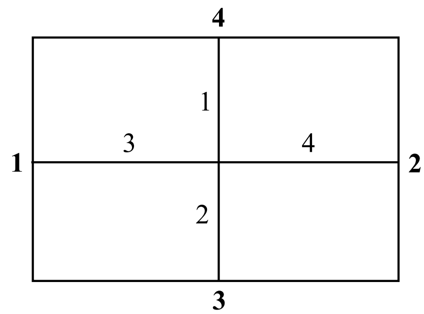


Figure 3.4: Two labelled coverings over P

Observe there are 2 coverings in Figure 3.4 superimposed over each other such that the covering elements labelled by 1 and 2 divide the space horizontally and those labelled by 3 and 4 split the space vertically. The resulting nerve is then a 3-dimensional simplicial complex satisfying the definition of simplicial game complex of $\mathcal{S}$ of the order 2, constructed from $\mathbf{m} = \{1, 2, 3, 4\}$ with $\mathbf{m_1} = \{1, 2\}$ and $\mathbf{m_2} = \{3, 4\}$. Following the notation from Definition 3.6 we set $\sigma = \{3\}$.

The closed star of σ, $\overline{St}(\sigma)$, is the simplicial complex shown in Figure 3.5

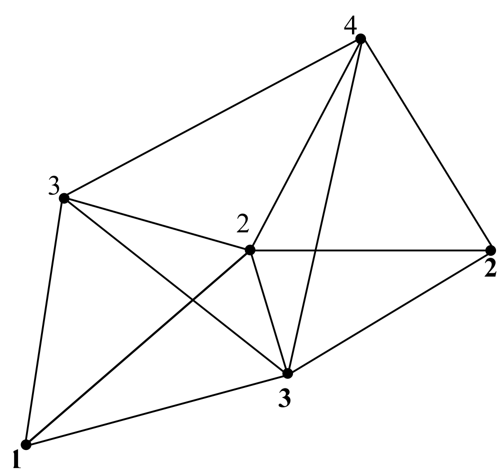


Figure 3.5: The closed star of σ

For clarity $\overline{St}(\sigma)$ consists of three 3-dimensional simplices with vertex sets $\{\mathbf{1}, \mathbf{3}, 2, 3\}$, $\{\mathbf{3}, 2, 3, 4\}$ and $\{\mathbf{2}, \mathbf{3}, 2, 4\}$ where the vertices in bold refer to their position as boundary faces. These vertices can all be found on the bottom 'edges' of the simplicial complex in Figure 3.5.

The highlighted simplices in Figure 3.6 show the open stars of σ and the open stars of all vertices from W_2.

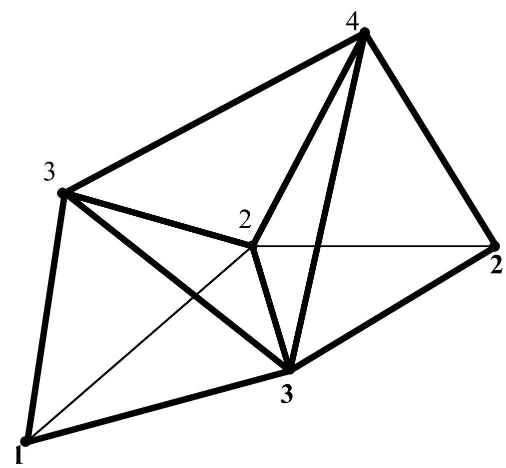


Figure 3.6: The open stars

Finally Figure 3.7 shows the closed star from Figure 3.5 minus the open stars highlighted in Figure 3.6. As is clear to see the result is a 1-dimensional simplicial complex satisfying the definition of $\mathcal{S}$ of the order 1 for $\mathbf{m} = \mathbf{m_1} = \{1, 2\}$.

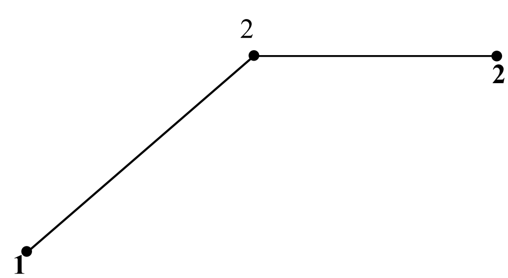


Figure 3.7: The sub-complex

(End Example)

This leads us to the following important Lemma.

Lemma 3.8
The subcomplex $\mathcal{S}'$, of a simplicial game complex $\mathcal{S}_k$, is a non-degenerate simplicial game complex of the order $k-1$ i.e., $\mathcal{S}' = \mathcal{S}_{k-1}$.

Proof. Observe for all $k \in \{2, \ldots, n\}$ we have $\mathcal{S}_{k-1} \subset \mathcal{S}_k$. In particular if σ_k is any face of $\partial\Sigma_{m_k}$ then $(\mathcal{S}_{k-1} * \sigma_k) \subset \mathcal{S}_k$ and consequently $\mathcal{S}_{k-1} \subset \overline{St}(\sigma_k)$. Further, if any vertex x belonging to the $\overline{St}(\sigma_k)$ is labelled by an element from $\mathbf{m_1} \cup \cdots \mathbf{m_{k-1}}$ it must belong to $\mathcal{S}_{k-1}$. Therefore it must be the case that $\overline{St}(\sigma_k)$ restricted to the set V_{k-1} is equal to $\mathcal{S}_{k-1}$ and we deduce $\mathcal{S}' = \mathcal{S}_{k-1}$. □

We now present a Lemma which will be key to our proof of Theorem 3.5 for non-degenerate simplicial game complexes.

Lemma 3.9
A sub-equilibrium $(m-2)$-dimensional simplex Λ belongs to the boundary of $\mathcal{S}$ iff there is an equilibrium simplex Λ' of S' such that $\Lambda = \Lambda' * \sigma$, where σ is the face of Σ_{m_n} with $\phi(\sigma) = \{m_{n-1}+1, \ldots, m_n - 1\}$.

Proof. Assume there is an equilibrium $(m_{n-1}-1)$-dimensional simplex Λ' of $\mathcal{S}'$ such that $\Lambda = \Lambda' * \sigma$, then clearly Λ is sub-equilibrium. Observe by the definition of $\mathcal{S}'$ this join is guaranteed to exist. We need to prove that Λ belongs to $\partial\mathcal{S}$. Indeed, since $\mathcal{S}$ is a non-ramified complex there is at least one vertex $x \notin v(\Lambda' * \sigma)$ such that $\Delta := \Lambda * \{x\}$ is a maximal simplex in $\mathcal{S}$. Note $\sigma = \Sigma_{\Delta,n}$. Since $W_{\Delta,n} \neq \emptyset$ and by construction $W_n \cap v(\Lambda' * \sigma) = \emptyset$ we arrive at $W_{\Delta,n} = \{x\}$. It now follows from the definition that Λ lies in the boundary.

Conversely, let Λ be a sub-equilibrium simplex belonging to $\partial\mathcal{S}$. By the definition of the boundary, there is a singleton $\{x\} = W_{\Delta,i}$ for some $i = \{1, \ldots, n\}$, where $\Delta := \Lambda * \{x\}$ is a maximal simplex in $\mathcal{S}$.

If $\phi(x) = m$, then $\{x\} = W_{\Delta,n}$. Then $\phi(\sigma) = \{m_{n-1}+1, \ldots, m_n - 1\}$ for $\sigma = \Sigma_{\Delta,n}$. Thus, for an appropriate Λ', the simplex Λ is of the required form $\Lambda = \Lambda' * \sigma$.

Suppose that $\phi(x) < m$, then there is another vertex, say y, in Λ with $\phi(y) = \phi(x)$. Clearly, y is a vertex in $\Sigma_{\Delta,i}$. Suppose first that $\phi(x) = t \leq m_{n-1}$. Then there are vertices, say $w_1, \dots, w_{m_i - m_{i-1} - 1}$ in Λ with $\phi(\{w_1, \dots, w_{m_i - m_{i-1} - 1}\}) = \{m_{i-1} + 1, \dots, t-1, t+1, \dots, m_i\}$. None of these vertices can be in $W_{\Delta,i} = \{x\}$, and can not all be in Σ_{m_i}, else $\{w_1, \dots, w_{m_i - m_{i-1} - 1}, y\}$ will be the set of all vertices of Σ_{m_i} , hence Σ_{m_i} is in S which is impossible. Therefore the supposition that $\phi(x) = t \leq m_{n-1}$ leads to a contradiction. It follows that $\phi(x) \in \{m_{n-1} + 1, \dots, m_n - 1\}$, thus $\{x\} = W_{\Delta,n}$. Then for $\sigma = \Sigma_{\Delta,n}$ we have $\phi(\sigma) = \{m_{n-1} + 1, \dots, m_n - 1\}$. Thus, for an appropriate Λ', the simplex Λ is of the required form $\Lambda = \Lambda' * \sigma$

□

To complete this section we give a final definition

Definition 3.10 (Graph G)
For simplicial game complex $\mathcal{S}$ let V be the set of all $(m-1)$-sub-equilibrium simplices and E the set of $(m-2)$-sub-equilibrium faces. Then define $G = (V, E)$ to be an undirected graph with vertex set V and set of edges E.

Observe an edge exists between a pair of vertices in G iff the corresponding simplices share a sub-equilibrium face in $\mathcal{S}$. Then the non-ramification property of $\mathcal{S}$ ensures the degree of G is less than or equal to 2. Therefore using the terminology from Lemma 2.22 a vertex in graph G is **normal** if it corresponds to an equilibrium simplex. A normal vertex is **extreme** if it is of degree 0. A non-normal vertex is called **extreme** if its degree is 1.

Remark
When $n = 2$, the graph G is equivalent to θ^m (minus the artificial equilibrium) as defined in Shapley's description of the Lemke-Howson algorithm [Shapley, 1974]. Consequently G can be thought of as a subgraph of $F_1 \times F_2$. The case $n > 2$ is discussed in Section 3.6.

3.5 Nash's Theorem for Simplicial Game Complexes

We now prove Nash's Theorem for non-degenerate simplicial game complexes $\mathcal{S}$.

Theorem 3.11 (Nash's Theorem for a Non-Degenerate Simplicial Game Complex)
Every non-degenerate simplicial game complex $\mathcal{S}$ contains a finite and odd number of equilibrium simplices.

Proof. We proceed by induction on n. In the case $n = 1$ we have $\mathbf{m} = \mathbf{m_1}$ and therefore $v(\phi(\Sigma_{m_1})) = \mathbf{m}$. Consequently $\phi(v(\mathcal{S})) = \phi(v(\Sigma_{m_1})) \cup \phi(W_1)$ and $\partial\mathcal{S} = \partial\Sigma_{m_1}$. For the simplicial complex $\mathcal{S}$ construct the graph G as defined in Definition 3.10. Observe the extreme vertices of G are those with degree 1 less than maximal. In particular the extreme vertices in G represent simplices with a sub-equilibrium face not belonging to a second simplex and therefore correspond to simplices with a sub-equilibrium face lying in the boundary of $\mathcal{S}$. Further if Λ is a sub-equilibrium boundary face of $\mathcal{S}$ then it must necessarily belong to a simplex which is not only represented in G but for which the associated vertex is extreme. Therefore the number of extreme vertices in G is equal to the number of sub-equilibrium simplices lying in $\partial\mathcal{S}$. Then since there is a unique sub-equilibrium face lying in $\partial\Sigma_{m_1}$ the graph G contains 1 extreme vertex. Using Lemma 2.22, we deduce the number of normal simplices in G is odd which by definition is equivalent to the number of equilibrium simplices in $\mathcal{S}$ being odd.

Remark
The simplicial game complex $\mathcal{S}$ maybe a simplex satisfying Sperner's Lemma, Lemma 2.21. Alternatively it maybe, for example, a triangulation of a torus with one open simplex removed.

For the induction step assume S', the simplicial complex of order $n-1$, contains a finite and odd number of equilibrium simplices. Then the remainder of the proof is almost identical to that seen for the base of induction.

For the simplicial complex of order n, $\mathcal{S}$, we have $\mathbf{m} = \mathbf{m_1} \cup \cdots \cup \mathbf{m_n}$. First observe the number of simplices in $\mathcal{S}$ is bounded and in particular the number of equilibrium simplices in $\mathcal{S}$ must be finite. For the simplicial complex $\mathcal{S}$ construct the graph G as defined in Definition 3.10. Then once again we observe the number of extreme vertices in G is equal to the number of sub-equilibrium simplices lying in $\partial\mathcal{S}$. According to Lemma 3.9 every sub-equilibrium face Λ in $\partial\mathcal{S}$ is uniquely representable as $\Lambda = \Lambda' * \sigma$, where Λ' is an equilibrium simplex in $\mathcal{S}'$, and σ is the face of Σ_{m_n} with $\phi(\sigma) = \{m_{n-1}+1, \ldots, m_n - 1\}$. It then follows that the number of extreme vertices in G is equal to the number of equilibrium simplices in $\mathcal{S}'$, and in particular must be odd. Finally, using Lemma 2.22 we can deduce the number of normal simplices in G is finite and odd and therefore so is the number of equilibrium simplices in $\mathcal{S}$. $\square$

3.6 Comparison to Lemke-Howson

To complete this Chapter we compare the method of identifying equilibrium simplices within $\mathcal{S}$ with the constructive procedure contained in [Lemke and Howson Jr, 1964], and as presented by Shapley [Shapley, 1974], to identify equilibrium situations in a non-degenerate bimatrix game.

Our proof has relied on the ability to construct a graph from our simplicial game complex, this is similar to the construction of F_i from $\tilde{P}_i$ as described in Chapter 1.7. With the details of Chapter 1.7 in mind we analyse the method to identify equilibrium points within our graph G and simplicial game complex $\mathcal{S}$.

We begin by identifying an extreme vertex of G and from this point we follow the complete connected path of edges and vertices which lead from it; where two nodes in G are distinct end points of the same edge if and only if the corresponding simplices in $\mathcal{S}$ share a sub-equilibrium face. Then in G either the path terminates with a unique normal (equilibrium) node (which maybe the starting node) or it is a path containing no normal nodes but where both end points are extreme nodes. The latter case is equivalent to following a path of sub-equilibrium simplices in $\mathcal{S}$ which begins and ends on the boundary of the simplicial game complex and consequently does not identify any equilibrium points of the game. However since the number of extreme nodes is odd, if you follow all paths which start from these extreme nodes you will eventually have to identify an equilibrium simplex. Observe that in fact you will have to try no more than half (rounded up since the number of extreme nodes is odd) of these paths as every path which ends in an extreme nodes eliminates two options from your list of starting points.

To allow comparison to Shapley, as we consider paths through the graph G we interpret them in terms of the simplices contained in the simplicial game complex $\mathcal{S}$. For this subsection we therefore use the following notation.

Notation

Let E_i represent a node in the graph G and Δ_i the corresponding simplex in the simplicial game complex $\mathcal{S}$

Within the graph G, to every node and edge assign the non-empty set of labels given to the simplices they represent in $\mathcal{S}$. Consider a extreme node E_1 and simplex Δ_1 with sub-equilibrium labelling. Then by definition this node/simplex is missing label m and will have a label from the set $\{1, \ldots, m-1\}$ repeated. Equivalently E_1 belongs to the set θ^m defined by Shapley. In $\mathcal{S}$ there will be two faces of Δ_1 where no two vertices are identically labelled; these will both be sub-equilibrium. However E_1 is assumed to be extreme and consequently one of the sub-equilibrium faces of Δ_1 belongs to the boundary of $\mathcal{S}$ and so does not belong to a second simplex. Recall the edges in G show simplices which share a sub-equilibrium face. Therefore E_1 is of degree 1 and there is a unique, sub-equilibrium, edge to follow. This edge also belongs to the set θ^m. Traversing this edge is equivalent to 'discarding' one of the repeated labels.

The edge will terminate with a node E_2 associated to simplex Δ_2 and this corresponds to the 'collection' of a new label. If Δ_2 is a sub-equilibrium simplex then the new label is a repeat of what is already present and there are two sub-equilibrium faces of Δ_2 corresponding to a node E_2 of degree 2. Since we have entered this node along one edge, our departure edge is uniquely determined. Traversing this edge (or passing through to the next simplex in $\mathcal{S}$ via the shared sub-equilibrium face)is equivalent to dropping one of the repeated labels. This process of dropping and collecting labels continues until E_k and Δ_k are completely labelled (or E_k is an extreme node). In this case there is no label to drop, and so no edge to depart along in G, and the path terminates.

Remark

By definition G is equal to the set θ^m minus the artificial equillibrium point.

This method of constructing a path is identical to that described Shapley.

There are however some differences between the overall constructive procedures. Shapley defined an artificial equilibrium which formed the starting point of a path which would guarantee to terminate with an equilibrium point. Currently this can only be replicated for simplicial game complexes of order 1. In this case a new vertex is added and connected to all vertices belonging to $\partial\mathcal{S}$. This new object is of dimension one higher than $\mathcal{S}$ and $\mathcal{S}$ can be identified as belonging to one of the faces, identical to as seen in Figure 1.17. Label this new vertex by m then we can identify a completely-labelled simplex lying outside of $\mathcal{S}$. From this simplex we remove the new vertex and we are left with a sub-equilibrium face belonging to $\partial\mathcal{S}$. In line with the proof presented for Theorem 3.11 this is the only sub-equilibrium face on the boundary of $\mathcal{S}$. Then, by the property of non-ramification, this face must belong to the simplex corresponding to the unique extreme point of G. Consequently following this path, as described above, must result in reaching an equilibrium situation of the game. In terms of G the new vertex is equivalent to adding a new completely labelled node and m almost completely labelled edges, each dropping one of the labels from $\{1, \ldots, m\}$.

Remark
We label the new vertex m to enable us to identify the unique sub-equilibrium face on the boundary and consequently a node in G.

Remark
The addition of the new vertex is equivalent to identifying the unique extreme node of G.

Observe the added equilibrium simplex is equivalent to the artificial equilibrium introduced by Shapley. However, in our case, the first choice of label to drop is fixed as we are currently bound to following sub-equilibrium paths, or those belonging to θ^m.

For simplices of larger order we can once again repeat this procedure, with the additional vertex joining to all vertices which belong to a boundary face of $\mathcal{S}$. However, this time we will be left with numerous additional equilibrium simplices to choose from to begin our path, each one taking us to a different extreme point of G. This leads us to the same problems discussed before as we have no way of knowing if our path will return to the boundary of $\mathcal{S}$ or produce an equilibrium simplex. We return to this again in Chapter 5 when we return to simplicial game complexes of the order 2.

Remark
For simplicial complexes of order greater than or equal to two, this procedure will produce a unique path precisely when there is a unique equilibrium simplex in the subcomplex as defined in Definition 3.6.

A final important observation. Let $\mathcal{S}$ be a simplicial game complex of the order n, and let every simplex in $\mathcal{S}$ represent a node, labelled identically to the simplex it represents.

Then two nodes are joined by an edge if the corresponding simplices share a face in $\mathcal{S}$. Denote this new graph by G^*. Then G is a subgraph G^* and G^* is comparable to (but not equal to) $F_1 \times F_2$. In particular G^* does not contain the artificial equilibrium point $\{0, 0\}$. Additionally, the construction of the boundary of $F_1 \times F_2$ (product of two simplices) differs to that of G^* (join of the boundary of n simplices) and this is the cause of problems in the cases where we are unable to identify a starting point for a path in G which will guarantee to terminate with an equilibrium situation. In later Chapters we will refer to these differences as **dimensional considerations**.

Chapter 4

Non-cooperative Games as Simplicial Complexes

We now return our focus to the motivation of this Thesis; Nash's Theorem for non-cooperative games. In this Chapter we begin by describing our interpretation of an N-player, non-cooperative game Γ and culminate with our proof of Nash's Theorem for non-degenerate games. We also use this Chapter to define our **generalisation** of Γ. Such generalised games will provide one of the most significant diversions when comparing our results to previously published work in this field.

4.1 Non-cooperative Games

We define an N-player non-degenerate, non-cooperative game Γ, with set of players $i = \{1, \ldots, N\}$. Each player i has a finite pure strategy set S_i with $|S_i| = l_i$ and mixed strategy set P_i. The expected payoff awarded to player i is given by the polylinear function G_i.

Recall $P^{(i)} := P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_N$ is the set of all possible strategy combinations of players $\{1, \ldots, i-1, i+1, \ldots, N\}$. An element from this set, $\mathbf{p^{(i)}}$, represents just one of these situations.

This **traditional**, or normal form, description of Γ will not be used explicitly in the original work contained in this Thesis. Instead, we use it as a foundation from which we construct an alternative representation of a game. Given our focus is Nash equilibria, for all players

$i = \{1, \ldots, N\}$ and for all situations $\mathbf{p^{(i)}} \in P^{(i)}$, we are only interested in the strategies from P_i which result in player i achieving his optimal payoff value. Using this as our motivation we now describe how we will represent and consider Γ.

From Chapter 1.3.4 we know the mixed strategy set P_i is a simplex of dimension $(l_i - 1)$. Therefore the set $P^{(i)}$ is the product space of $N - 1$ simplices and P is the product space of all N simplices, both can be considered as a bounded subset of $\mathbb{R}^l$ for appropriate $l \in \mathbb{N}$. This will be an important aspect of our representation of Γ.

Example 4.1
Let Γ be a 3-player non-degenerate dyadic game. Then for every $i = \{1, 2, 3\}$ let $p_i = (x_i, 1 - x_i)$ and observe the corresponding simplex P_i is 1-dimensional. Assume without loss of generality $i = 1$ then $P^{(1)} = P_2 \times P_3$ and $P = P_1 \times P_2 \times P_3$ are shown in Figure 4.1.

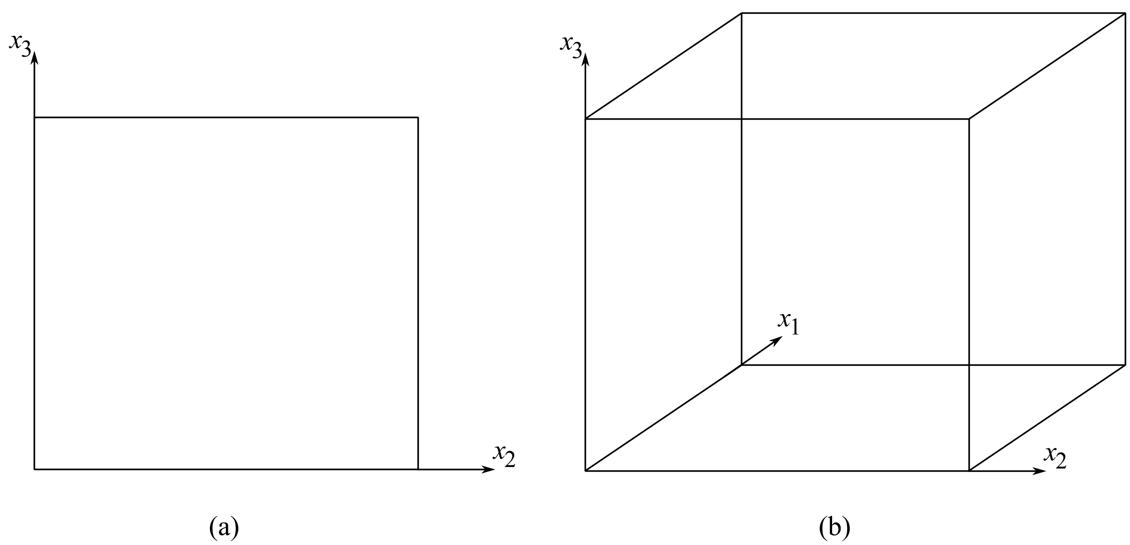


Figure 4.1: (a) $P^{(1)}$ and (b) P as a geometric object

(End Example)

Recall for every $i = \{1, \ldots, N\}$ and $s_i^j \in S_i$, $\quad (\mathbf{p}, s_i^j) := (p_1, \ldots, p_{i-1}, s_i^j, p_{i+1}, \ldots, p_N)$.

For every situation $\mathbf{p} \in P$ the payoff function G_i allows the identification of $s_i^j \in S_i$ such that

$$G_i(\mathbf{p}, s_i^j) \geq G_i(\mathbf{p}, s_i^{j'}) \qquad \forall\, s_i^{j'} \in S_i \tag{4.1}$$

For each player i, let Z_i be the non-empty subset of strategies from S_i which satisfy (4.1). Then, in particular, for each situation $\mathbf{p^{(i)}}$, Z_i is the non-empty subset of pure best responses from S_i.

$$Z_i \;:=\; \{s_i^j \,|\, G_i(\mathbf{p}, s_i^j) \geq G_i(\mathbf{p}, s_i^{j'}) \;\; \forall\, s_i^{j'} \in S_i\} \;\subset\; S_i \tag{4.2}$$

Remark

Recall the definition of the support Y_i, Definition 1.6, then a mixed strategy is a best response if $Y_i \subseteq Z_i$.

Alternatively for every $s_i^j \in S_i$ the (possibly empty) set of strategies from $P^{(i)}$ such that $s_i^j \in Z_i$ can be identified. Therefore for all situations $\mathbf{p} \in P$ and pure strategies $s_i^j \in Z_i$ define

$$\bar{B}_i(s_i^j) := \{\mathbf{p^{(i)}} \,|\, G_i(\mathbf{p^{(i)}} \times \{s_i^j\}) \geq G_i(\mathbf{p^{(i)}} \times \{s_i^{j'}\}) \;\; \forall\, s_i^{j'} \in S_i\} \tag{4.3}$$

Where we recall $\mathbf{p^{(i)}} \times \{s_i^j\}$ defines a situation $\mathbf{p} \in P$ for all $\mathbf{p^{(i)}} \in P^{(i)}$ and $s_i^j \in S_i$.

Consider $\bar{B}_i(s_i^1) \cup \cdots \cup \bar{B}_i(s_i^{l_i})$. Then since the payoff function G_i ensures each player i must have at least one optimal strategy across $P^{(i)}$ there exists at least one subset $\bar{B}_i(s_i^j)$ such that $\bar{B}_i(s_i^j) \neq \emptyset$. Further since for every $\mathbf{p^{(i)}} \in P^{(i)}$ player i must be able to identify his optimal strategy each $\mathbf{p^{(i)}}$ must belong to at least one subset. Therefore we have

$$P^{(i)} \;=\; \bigcup_{s_i^j \in S_i} \bar{B}_i(s_i^j) \tag{4.4}$$

and as such the sets $\bar{B}_i(s_i^j)$ form a finite covering over $P^{(i)}$.

Example 4.2

Continuing from Example 4.1. By definition each player's payoff functions G_i are polylinear. Therefore, by following the discussion above, for every $i = \{1, 2, 3\}$ the covering over $P^{(i)}$ takes the form seen in Figure 4.2. Each element of the covering is labelled by the set $\bar{B}_i(s_i^j)$ it represents.

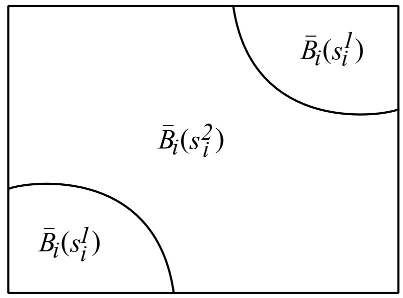


Figure 4.2: Dyadic game covering over $P^{(i)}$

(End Example)

We observe in Figure 4.2 the covering element defined by $\bar{B}_i(s_i^1)$ is disconnected. To ensure our description of a game is clear we insist each element of the covering is a connected set. In particular we redefine each element of the covering over $P^{(i)}$ to be a connected component of a set $\bar{B}_i(s_i^j)$. Denote such connected components by $B_i(s_i^j)$. Then $B_i(s_i^j)$ is a maximal connected subset of $\bar{B}_i(s_i^j)$. Observe the property of polylinearity inherent in the payoff function ensures the number of such connected components is finite.

We refer to the subsets $B_i(s_i^j)$ as **covering elements** of $P^{(i)}$.

Remark
Our covering elements are equivalent to the sets P_i^k defined in (1.43) of Chapter 1.7. In particular, these sets identify pure best responses for one player given a situation from $P^{(i)}$.

4.2 Generalised Games

Examining the properties of the covering defined over $P^{(i)}$ leads us to a generalisation of the game Γ. By definition the sets $B_i(s_i^j)$ contain the situations from $P^{(i)}$ for which pure strategy $s_i^j \in S_i$ yields an optimal payoff for player i. However the actual payoff value received by player i does not feature in the construction of the covering. Instead we have only used the payoff functions to order each player's pure strategies with regards to preference. Consequently instead of a payoff function, we only require each player i to define a **total order** over his set of pure strategies.

By definition, the axioms of a total order on a set A state for elements $a, b \in A$, if $a \geq b$ and $b \geq a$ then $a = b$. However this is not quite the definition we require. Suppose player i is indifferent between pure strategies s_i^1 and s_i^2, then it is natural to expect a total order on the set of pure strategies to reflect this. It is clear that because $s_i^1 \neq s_i^2$ a total order applied to this set, as described above, would not achieve our desired outcome. To rectify this problem we define a surjective map from the set of mixed strategies into a finite set of totally ordered elements K, such that strategies of equal preference are mapped to the same element. Naturally, the total order on K together with the surjective map will reflect the preferences of player i.

Definition 4.3 (Total Order)
For every player $i = \{1, \ldots, N\}$ define the surjective function $\succ_i$: $P_1 \times \cdots \times P_N \mapsto K_i$, where K_i is finite, totally ordered set, such that

1. *For every fixed $\mathbf{p^{(i)}} \in P_i$ the maximum with respect to $\succ_i$ of $\mathbf{p} = \{p_1, \ldots, p_N\}$ over p_i is attained at an element of S_i.*

2. *If the maximum is attained at $s_i^1, \ldots, s_i^l \in S_i$ then the maximum must also be reached across the convex hull of these points and equivalently across the corresponding face in P_i.*

Definition 4.4 (Generalised Game)
In an N-player non-cooperative game Γ for all $i = \{1, \ldots, N\}$ replace each payoff function G_i with a total order $\succ_i$. The resulting game is a non-cooperative generalised game*, or* generalised game*, and is denoted by Γ^*.*

Remark
We will remain using the term 'total order' as this provides a clear and intuitive understanding of our requirements.

Within our generalised games the notion of a player's pure best response and support are equivalent to its definition in the traditional game setting. Then our constraints on a total order, Definition 4.3, is equivalent to requiring that for every situation in $P^{(i)}$ player i's best response is a pure strategy. The support of a mixed strategy for player i is once again the non-empty subset of pure strategies which are assigned a strictly positive value and consequently (at optimal strategies) $Y_i \subseteq Z_i \subseteq S_i$ still holds.

As before, we construct a covering over $P^{(i)}$ which can be formally expressed as connected sets, indicating best responses, of the kind

$$B_i(s_i^j) \;:=\; \{\mathbf{p^{(i)}} \mid (\mathbf{p^{(i)}} \times \{s_i^j\}) \succ_i (\mathbf{p^{(i)}} \times \{p_i\}) \quad \forall\, p_i \in P_i\} \tag{4.5}$$

for all $s_i^j \in S_i$.

Suppose the best pure responses for player i, for a given situation $\mathbf{p^{(i)}} \in P^{(i)}$, are contained in the set Z_i. Then $\mathbf{p^{(i)}} \in B_i(s_i^j)$ for all $s_i^j \in Z_i$. It is natural to except the intersection of covering elements to represent situations where a player is indifferent between his pure strategies and so will consider mixing them. Therefore property 2 of Definition 4.3 ensures the sets $B_i(s_i^j)$ are closed, bounded and intersect properly.

Remark
The finite condition on the set K_i from Definition 4.3 is equivalent to the generalised game containing a finite number of covering elements. In particular a given pure strategy can only be optimal a finite number of times.

Example 4.5
Continuing from Example 4.2 we provide an intuitive example of this. Replacing the poly-linear payoff functions G_i with a total orders $\succ_i \; \forall\, i = \{1, 2\}$ can produce the generalised coverings seen in Figure 4.3.

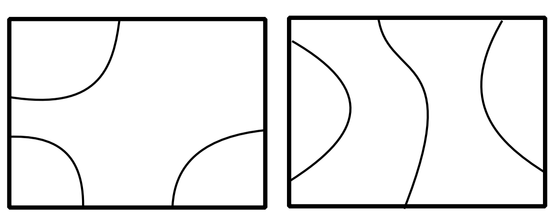


Figure 4.3: Generalised dyadic game coverings

(End Example)

The following three examples of total orders (defined as payoff functions) all illustrate Definition 4.3, either as a counter-example or as an example satisfying this definition.

All three examples define a dyadic bimatrix game with mixed strategies given by $(x_1, 1-x-1)$ and $(x_2, 1-x_2)$ for player's 1 and 2 respectively. As we have seen in Example 1.15 the payoff functions will be given in terms of the independent strategy only, namely x_1 and x_2. Since x_1 and x_2 are probability distributions we require $x_i \in [0,1]$ for $i = \{1,2\}$. In the examples which follow the range of allowed values, $[a,b]$, is larger than this. However the functions and domains can be rescaled to satisfy the required conditions so does this not cause any problems (we will work with the original formulation). Note the 'pure strategy' set for each player is defined when x_i takes the extreme values of the set i.e., at $x_i = a$ and $x_i = b$.

Example 4.6

Let Γ be a dyadic bimatrix game with mixed strategies $(x_1, 1-x_1)$ and $(x_2, 1-x_2)$ respectively. Assume the payoff functions are given by

$$\begin{aligned} G_1(x_1,x_2) &= -x_1 \cdot x_2 + (x_2)^2 \qquad \text{for } x_1,\, x_2 \in [-1,1] \\ G_2(x_1,x_2) &= x_1 \cdot x_2 - (x_2)^2 \qquad \text{for } x_1,\, x_2 \in [-1,1] \end{aligned} \tag{4.6}$$

Consider $G_1(x_1,x_2)$ then, for all values of x_2, this function is linear with respect to player 1. In particular for any value $x_2 \in [-1,1]$, $G_1(x_1,x_2)$ must achieve its maximum when either $x_1 = -1$ or $x_1 = 1$. If the maximum is achieved at both points then the gradient of the function is 0 and consequently all mixed strategies must also be best responses. Consequently G_1 satisfies Definition 4.3.

Now look at $G_2(x_1,x_2)$. Fix $x_1 = 0$ then $G_2(0,x_2) = -(x_2)^2$ and the resulting graph is a negative quadratic function which achieves its maximum at $x_2 = 0$. Therefore this fails Definition 4.3 and this example does not represent a genearlised game.

To graphically understand the necessity of the violaed condition, consider the best response correspondences for these two payoff functions, given in Figure 4.4.

Once the appropriate scaling of the payoff functions has been done, the best response correspondence of (a) is of the standard form seen in Chapter 1.3.5. Consequently the covering over P_2 (the interval $[-1,1]$) using sets $B_1(s_1^i)$ is finite. In contrast, the best response correspondence of (b) is a straight line of constant non-zero gradient. As a result

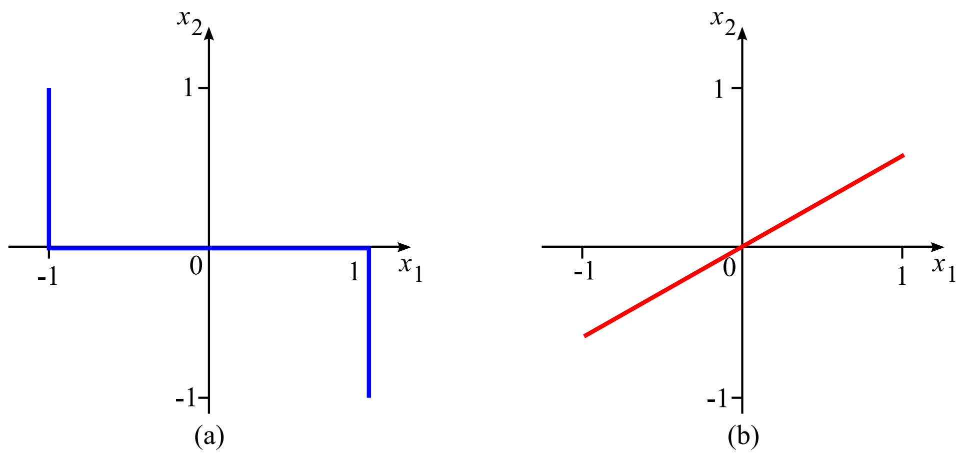


Figure 4.4: Best response correspondences for $(a)\, G_1$ and $(b)\, G_2$ for Example 4.6

of this we cannot determine a finite covering of P_1 as there is a different best response to every element from $[-1, 1]$. This contradicts Definition 4.3.

Remark
The two best response correspondences seen in Figure 4.4 do have a non-empty intersection and therefore an equilibrium situation must exist. Therefore it maybe possible to find an alternative definition to Definition 4.3 to replace the payoff functions which is less restrictive. However this, or understanding why this example has an equilibria, will not be considered in this Thesis.

(End Example)

Example 4.7
This example considers the reverse of the example given above so is a bimatrix game described as in Example 4.6 with payoff functions as follows

$$\begin{aligned} G_1(x_1, x_2) &= x_1 \cdot x_2 - (x_2)^2 \qquad \text{for } x_1,\, x_2 \in [-1, 1] \\ G_2(x_1, x_2) &= -x_1 \cdot x_2 + (x_2)^2 \qquad \text{for } x_1,\, x_2 \in [-1, 1] \end{aligned} \tag{4.7}$$

Once again G_1 is linear with respect to player 1's strategy and G_2 is a quadratic with respect to player 2. The difference this time is that G_2 is always maximal at the end points, i.e., when $x_2 = -1$ or $x_2 = 1$ and therefore the first part of Definition 4.3 holds for both payoff functions. However consider $G_2(0, x_2) = (x_2)^2$. Then $G_2(0, x_2)$ achieves its maximum when $x_2 = 1$ and $x_2 = -1$ but not for any $x_2 \in (-1, 1)$. The corresponding best response correspondence for G_2 is shown in Figure 4.5.

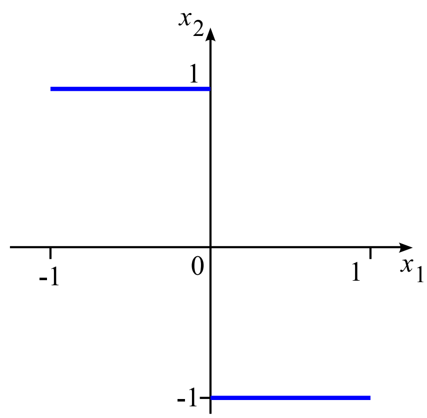


Figure 4.5: Best response correspondence for G_2 in Example 4.7

In Figure 4.3 at $x_1 = 0$ there is a discontinuity. In terms of the covering this equivalent to the covering elements not intersecting properly; i.e., an intersection between covering elements would not represent an indifference between strategies 1 and -1 and any combination of them. Hence the appropriate covering is once again unable to be constructed and this example fails the second part of Definition 4.3.

(End Example)

Example 4.8

Consider the bimatrix game Γ defined by payoff functions

$$\begin{aligned} G_1(x_1, x_2) &= x_2 \cdot (x_2 - 1) \cdot (x_2 - x_1) \qquad \text{for } x_1,\, x_2 \in [-1, 2] \\ G_1(x_1, x_2) &= -x_1 \cdot (x_2)^3 \qquad \text{for } x_1,\, x_2 \in [-1, 2] \end{aligned} \tag{4.8}$$

Then the best response correspondences for both functions are given in Figure 4.6

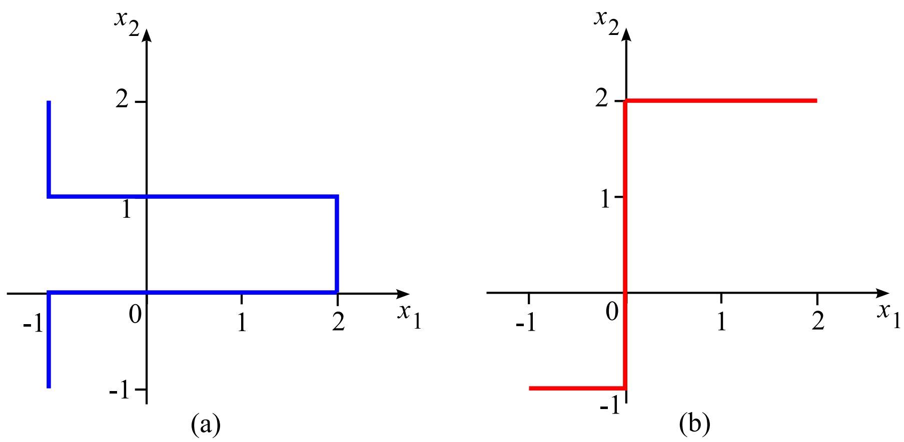


Figure 4.6: Best response correspondence for $(a)G_1$ and $(b)G_2$ in Example 4.8

The first thing we observe in Figure 4.6 is the best response correspondence in (a) is a generalised form to those seen previously. By definition, the best response correspondence in (a) shows player 1's best responses to all situations from P_2. As such it is easy to check that for all situations from P_2 the best response correspondence includes at least one of $x_1 = -1$ or $x_1 = 2$ and hence the first condition of Definition 4.3 is satisfied. Now observe in (a) there are two (disconnected) line segments at $x_1 = -1$ and one at $x_1 = 2$ whose union is equal to $[-1, 2]$, or player 2's mixed strategy simplex P_2. Therefore a finite covering over P_2 can be constructed. Finally there are no discontinuities. So for the two strategies from P_2 where player 1 is indifferent between his pure strategies, player 1 is also indifferent between any of his mixed strategies. Hence Definition 4.3 is completely satisfied with respect to G_1.

The best response correspondence (b) is of the form traditionally seen and in any case repeating the analysis above will determine that G_2 also satisfies Definition 4.3. Consequently this example is of a generalised game and by observation we see this game has 1 pure strategy equilibrium point and 2 mixed strategy equilibria situations.

(End Example)

In our generalised games the definition of a Nash equilibrium situation is a natural extension of Nash's original definition as given in Definition 1.10.

Definition 4.9 (Nash Equilibria for Generalised Games)
A situation $\mathbf{p} = (p_1, \ldots, p_i, \ldots, p_n) \in P$ *is a* Nash equilibrium situation *if*

$$\mathbf{p} \succ_i (p_1, \ldots, p_{i-1}, p_i', p_{i+1}, \ldots, p_N) \tag{4.9}$$

for every $i = \{1, \ldots, N\}$ *and every* $p_i' \in P_i$.

Before we state Nash's Theorem for generalised games we introduce the (preliminary) conditions of non-degeneracy for Γ^*. In the traditional game theory definition, Γ is said to be non-degenerate if the intersection of its best response correspondences is transverse. Observe in Γ segments of each best response correspondence can be interpreted as the intersections between covering elements, defined with respect to one player, over P. Therefore we require the intersection of the covering elements $B_i(s_i^j)$ be transverse and thus each covering element must be a smooth manifold.

Then Nash's Theorem for generalised games is as follows.

Theorem 4.10 (Nash's Theorem for Generalised Games)
Within each non-cooperative generalised game Γ^* *there exists an equilibrium situation. When* Γ^* *is non-degenerate the number of equilibrium situations is finite and odd.*

For a situation $\mathbf{p} \in P$ let Z_i be the set of pure best responses for player i. Then for all $i = \{1, \ldots, N\}$ replacing the order $\succ_i$ in Definition 4.9 with the polylinear payoff function G_i we arrive at:

$$G_i(\mathbf{p}, s_i^j) \geq G_i(\mathbf{p}, s_i^{j'}) \qquad \forall \;\; s_i^j \in Z_i, \;\; s_i^{j'} \in S_i \tag{4.10}$$

which is equivalent to equation (1.25) and thus to Definition 1.10. Additionally it is clear the payoff functions G_i define a total order on the strategies in S_i with regards to the situations in $P^{(i)}$. Finally we have seen the characteristics which define Γ to be (non)-degenerate impose the same property on Γ^* (we return to this later). Therefore the traditional description of the game Γ is contained in our generalised game Γ^*. In particular any result which is true for Γ^* will also hold for Γ. Consequently we will prove Theorem

4.10, Nash's Theorem for Γ^*, but in doing so we will have automatically constructed a proof for Theorem 1.13, the traditional statement of Nash's Theorem.

Remark

Using a total order $\succ_i$ will define games of an abstract nature which are not required to produce explicit payoff values. Should this property be necessary then we can revert back to using a payoff function providing it defines a total order. However such functions are no longer required to be polylinear.

The coverings over each $P^{(i)}$ only provide information about player i's optimal strategies. Recall all realisable situations from the generalised game Γ^* are contained within P and by definition $P^{(i)} \subset P$ for all $i = \{1, \ldots, N\}$. Therefore for each player i and all $s_i^j \in S_i$ consider $B_i(s_i^j) \times P_i$, then the union of all such sets must form a covering of P. Repeating for all players will result in N distinct coverings of P. Observe the collection of all N coverings will contain all optimal strategies for all player's. We therefore redefine the notation $B_i(s_i^j)$ as follows.

Definition 4.11 (Covering Element)
For all $i \in \{1, \ldots, N\}$ and $s_i^j \in S_i$ let $B_i(s_i^j)$ denote a connected component of $\bar{B}_i(s_i^j) \times P_i$. Then the union of all $B_i(s_i^j)$, for all connected components of $\bar{B}_i(s_i^j)$, will form a covering of P. The sets $B_i(s_i^j)$ are the covering elements *of P with respect to player i.*

Remark

Recall from Definition 4.3 there will be a finite number of covering elements.

Example 4.12

Continuing from Example 4.2, Figure 4.7 shows the covering given in Figure 4.2 extended to the space P. Observe how the intersection of each pair of covering elements forms a cylinder.

Figure 4.8 then shows all 3 player's coverings over the space P. Each of these coverings take the form seen in Figure 4.7 and all are pairwise orthogonal.

For clarity Figure 4.9 shows the intersections around the vertex $(1, 0, 1)$ from Figure 4.8 in more detail where (a), (b) and (c) show the intersection of cylinders associated to player's 1 and 2, 1 and 3 and 2 and 3 respectively while (d) is the intersection of all 3 cylinders.

(End Example)

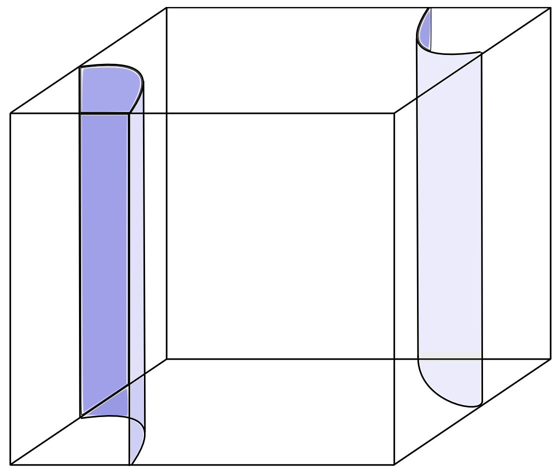


Figure 4.7: Dyadic game covering over P for player 3

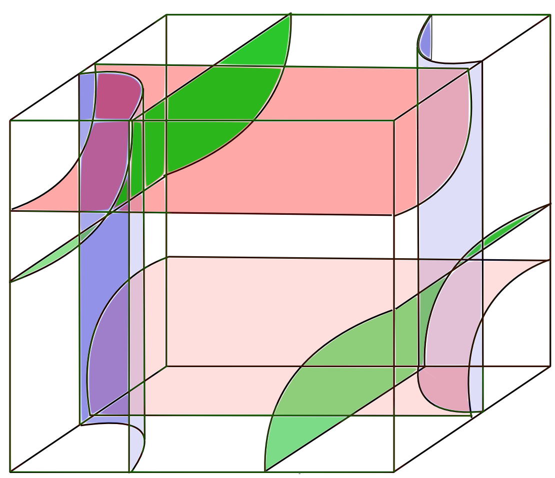


Figure 4.8: Dyadic game covering over P for all 3 player's

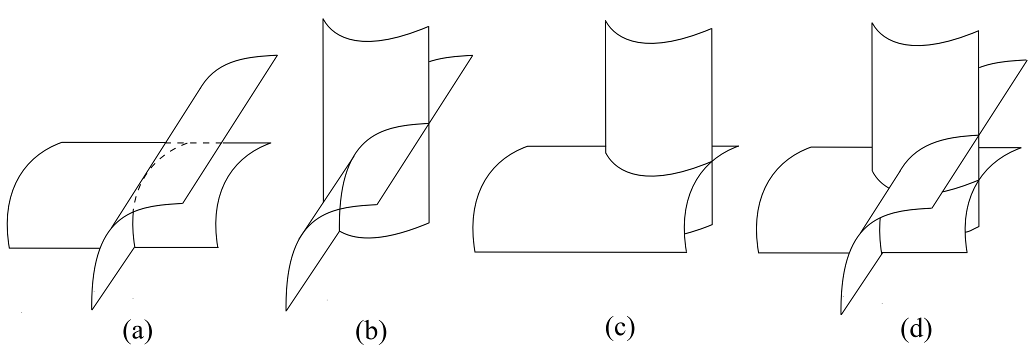


Figure 4.9: Intersection of covering elements

By definition of the covering, the relative positions of all covering elements will determine player i's optimal strategies for all situations $\mathbf{p}^{(\mathbf{i})} \in P^{(i)}$. As such it is the relative positions of the covering elements which will help determine equilibrium situations of the game. This information can be represented by constructing the **nerve** of the covering, (Definition 2.11). The red (bold) lines in Figure 4.10 illustrate the nerve of an arbitrary covering.

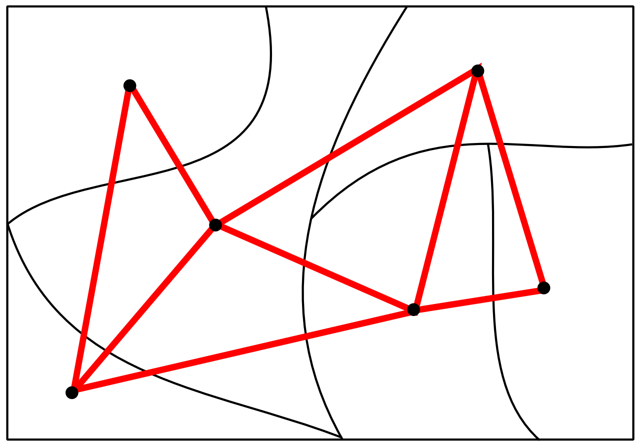


Figure 4.10: Constructing a nerve

Remark
The nerve of a covering can be considered the coverings **dual**.

This construction of the nerve only considers the optimal strategies for all players and does not provide any information regarding the actual strategy played by each player at that point. This is determined by the position of the intersection of the covering elements in P. As such the support for each player, for the situation represented by the intersection, can be determined by identifying the boundary faces (if any) the intersection belongs to.

Let $\mathbf{p} \in P$ be a situation lying away from the boundary, then $\mathbf{p}$ represents a totally mixed strategy situation for all player's. As such the support for every player $i = \{1, \ldots, N\}$ is S_i. Now suppose $\mathbf{p}$ belongs to boundary face Λ. If $\dim \Lambda = \dim P - 1$ then there exists a unique $i \in \{1, \ldots, N\}$ such that the support for player i is of size $(l_i - 1)$ while the support for all other player's $j = \{1, \ldots, i-1, i+1, \ldots, N\}$ is once again S_j. If $\dim \Lambda < \dim P - 1$ then Λ is the result of multiple faces (of dimension equal to $\dim P - 1$) intersecting. Consequently Λ corresponds to the situations from P where the support for some (possibly all) player's is strictly less than maximal. The number of pure strategies assigned 0 in $\mathbf{p}$ increases until $\mathbf{p}$ represents a vertex of P. At such points all players are selecting a pure strategy and each player has a support of size 1. It is therefore clear the inclusion of the boundary faces in the construction of the nerve will provide the 'missing information'.

Remark

We are referring to the support Y_i and not the set of pure best responses Z_i as we are not considering optimal strategy choices.

The boundary faces are represented in the nerve in the same manner as the covering elements. In particular each boundary face is assigned an arbitrary point of representation which will corresponds to a vertex in the nerve. Two vertices then share an edge if the corresponding boundary faces have a non-empty intersection. It is then natural to connect vertices corresponding to covering elements and boundary faces by an edge should the associated covering elements and boundary faces have a non-empty intersection.

Figure 4.11 illustrates the nerve of a simple and arbitrary covering over a bounded set in $\mathbb{R}^2$. The vertices labelled by 1,2,3 are the representative points of the covering elements, and those labelled by a,b,c,d are the ones for the boundary faces. Note the boundary face labels have been marked away from the boundary for clarity. The red (bold) lines then form the resulting nerve.

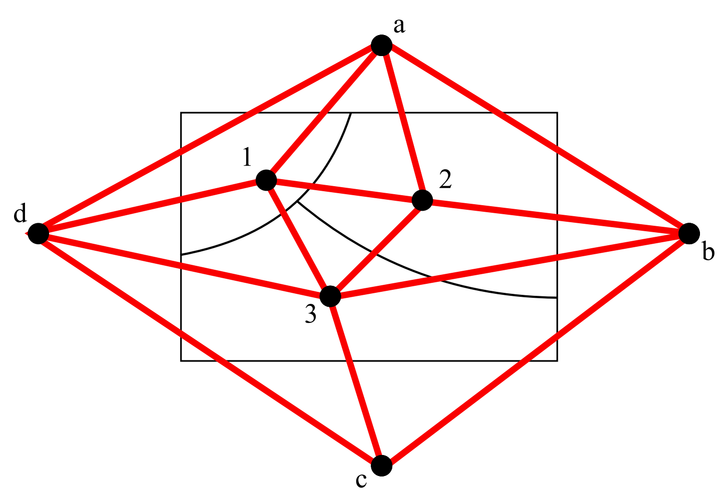


Figure 4.11: The nerve of a bounded covering

By definition the covering elements $B_i(s_i^j)$ over the bounded space P are closed connected subsets. There are also finitely many of them and hence the resulting nerve will contain a finite number of nodes. However we cannot assume their intersections will behave in a predictable way and we may have the situation where a subset of covering elements intersect with multiple dimensions or in multiple disjoint segments. The following example shows a simple example of this second scenario.

Example 4.13
Let Γ^* be a 3 player generalised game with pure strategy sets S_1, S_2 and S_3 where $|S_1| = 3$ and $|S_2| = |S_3| = 2$. Let a mixed strategy for player 1 be given by $(x_1^1, x_1^2, x_1^3) = (x_1^1, x_1^2, 1 - x_1^1 - x_1^2)$, and those for player 2 and 3 by $(x_2^1, 1 - x_2^1)$ and $(x_3^1, 1 - x_3^1)$ respectively. For this example each player is interested in minimising his outcome and therefore the result of each game is interpreted as a 'cost'. As such the functions determining outcome are referred to as cost functions.

Define the cost function for player 1, which is to be minimized and not maximized, by

$$G_1 : P_1 \times P_2 \times P_3 \quad \mapsto \quad x_1^1 \left(x_2^1 - \frac{1}{4}\right)\left(x_3^1 - \frac{3}{4}\right) \; + \; x_1^2(x_2^1 - x_3^1) \; + \; x_1^3(x_3^1 - x_2^1) \tag{4.11}$$

Restricting player 1's input to a pure strategy produces the following three restricted payoff functions:

$$G_1^1 \quad = \quad G_1(\mathbf{p}, s_1^1) : \{s_1^1\} \times P_2 \times P_3 \;\mapsto\; \left(x_2^1 - \frac{1}{4}\right)\left(x_3^1 - \frac{3}{4}\right) \tag{4.12}$$

$$G_1^2 \quad = \quad G_1(\mathbf{p}, s_1^2) : \{s_1^2\} \times P_2 \times P_3 \;\mapsto\; (x_2^1 - x_3^1) \tag{4.13}$$

$$G_1^3 \quad = \quad G_1(\mathbf{p}, s_1^3) : \{s_1^3\} \times P_2 \times P_3 \;\mapsto\; (x_3^1 - x_2^1) \tag{4.14}$$

Then for each situation in $P^{(1)}$ we can identify which of the three functions given by equations (4.12), (4.13) and (4.14) results in the optimal payoff for player 1. The covering representing the optimal strategies for player 1 is shown in Figure 4.12 where each covering element is labelled by the restricted payoff functions it represents.

We observe the covering elements representing pure strategies s_1^2 and s_1^3 being optimal for player 1 intersect in two distinct segments of equal dimension. The red (bold) lines in Figure 4.13 show the nerve, a simplicial complex, of the covering given in Figure 4.12.

First observe there is just one edge in the nerve which represents the intersection of covering elements labelled in Figure 4.12 by G_1^2 and G_1^3; this is despite this intersection occurring in

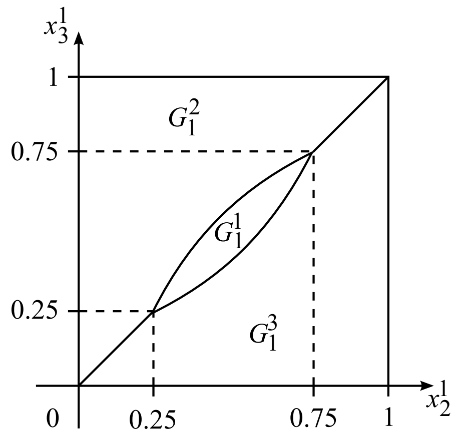


Figure 4.12: Covering of $P^{(i)}$

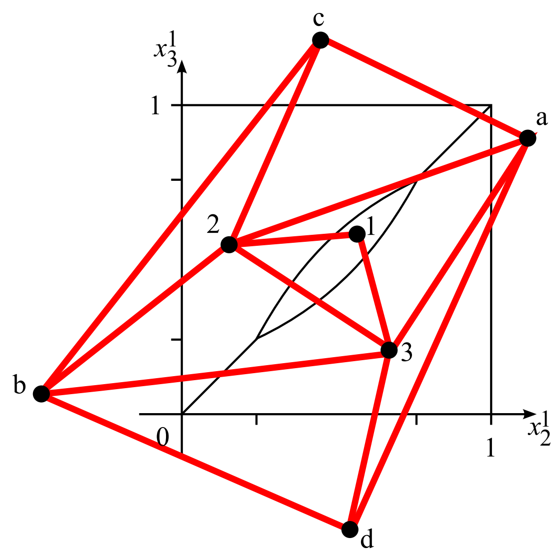


Figure 4.13: Nerve of the covering shown in Figure 4.12

2 disjoint components. As a direct result of this, the face $\{2, 3\}$ belongs to three simplices and the nerve fails the conditions of non-ramification (property 2 of Definition 2.7). This may cause complications with the definition of a non-degenerate nerve, Definition 2.19.

Remark

For clarity in representation we are considering the covering over $P^{(i)}$ and not P. However it is clear the same intersection problems will be present when the covering is extended and represented over P instead.

(End Example)

We return to the problem arising in Example 4.13 later in the Chapter, but for now we consider the general geometric form the nerve takes. Observe at every point in the covering there must be N covering elements present. If each player has a unique covering element over P, so every player's optimal strategy is a single pure strategy, the nerve will contain N vertices. These vertices must necessarily be affinely independent, and since all covering elements have a non-empty intersection, in the nerve an edge must exist between all pairs. Consequently the nerve at this point is a simplex of dimension $(N-1)$. Now suppose at a given point of intersection there are a total of k covering elements and boundary faces. Since each of the k covering elements and boundary faces intersect with the remaining $k-1$ covering elements and boundary faces, at the corresponding point in the nerve each pair of vertices must share an edge and thus is a simplex of dimension $k-1$. If there are multiple points of intersection across P then observe the set of covering elements and boundary faces at each intersection cannot be disjoint. Consequently, in the nerve, the resulting simplices have vertices in common and so the complete nerve is a simplicial complex of dimension $\bar{k}-1$ where $\bar{k}$ is the maximum number of covering elements and boundary faces intersecting at any one point.

Remark

Observe the nerve of a simplex is the original simplex. This is illustrated in Figure 4.14 where the red (bold) lines indicate the nerve

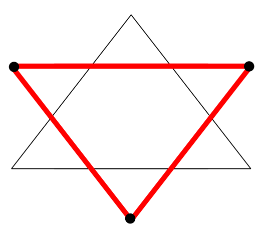


Figure 4.14: The nerve of a 2-dimensional simplex

Consider the bimatrix game where $P = P_1 \times P_2$. Then the nerve of P must contain the boundaries of P_1 and P_2, denoted by ∂P_1 and ∂P_2 respectively. Further in P each boundary face of P_1 is joined to every boundary face of P_2 and this property must be maintained in the nerve. Therefore the nerve of P is $\partial P_1 * \partial P_2$. This extends to any finite size game and in particular the nerve of P for the N-player game Γ^* is equal to $\partial P_1 * \cdots * \partial P_N$.

Notation

For the N-player non-cooperative generalised game Γ^* denote the nerve constructed from the N-coverings over P, including boundary, by η.

4.3 Labelling Procedure

In order to identify each simplex in η and to associate it to the situation it represents from the game Γ^*, without referring back to the orders $\succ_i$, we assign a unique label to each vertex. Since each vertex is either associated to a covering element (defined by an order) or a boundary face we define a labelling procedure for these two distinct sets. Each vertex in η then shares the same label as the covering element or boundary face it represents. Clearly these labels should reflect strategies played and therefore the set of all labels must have a natural bijective correspondence to $\{S_1, \ldots, S_N\}$. In light of this, and for clarity, we use the pure strategies as the set of labels.

The covering elements $B_i(s_i^j)$ defined by $\succ_i$ are naturally labelled by s_i^j, player i's optimal strategy across the set under consideration.

The labelling of the vertices associated to the boundary faces is not as trivial. By definition the number of independent variables across P is equal to the sum of all independent variables from P_i for all $i \in \{1, \ldots, N\}$. Therefore situations which do not fall on the boundary will involve all pure strategies and thus represent totally mixed strategy situations for all N player's. The dimension of the boundary faces are $(\dim P - 1)$ and as such contain one less independent variable than P. In particular each boundary face represent the situations from P which use all bar one of the independent strategies. That is, there is one pure strategy which does not feature in the support of any mixed strategy across the entire face in question and therefore is assigned the probability zero. The boundary face is then labelled by this pure strategy.

For simplex P_j the set of vertices correspond uniquely to all pure strategy situations available to player j. Therefore in the product space P the vertices are the situations where all players assign probability 1 to one of their pure strategies. It is clear each vertex in P corresponds to a unique vertex from P_i for all $i \in \{1, \ldots, N\}$. All combinations of vertices from the simplices $P_i \in P$ must be used and therefore the vertices in P represent all pure strategy situations uniquely.

Consider a one dimensional edge in P then this is governed by a single equation in two variables, namely $\; x_j^q + x_j^{q'} = 1 \;$ for $\; x_j^q,\, x_j^{q'} \in p_j,\; q \neq q' \;$ and for some $j \in \{1, \ldots, N\}$. All other probability distributions must remain the same at the two vertices and across the entire edge. We can assume without loss of generality that at vertex 1 we will have $x_j^q = 1, x_j^{q'} = 0$ and $x_j^q = 0, x_j^{q'} = 1$ at vertex two. Therefore the edge under consideration represents the mixed strategy situation where player j assigns non-zero probabilities to pure strategies s_j^q and $s_j^{q'}$ only with the remaining $(N-1)$ players choosing the pure strategies defined at the vertices. Since for all players $i = \{1, \ldots, N\}$ $\quad P_i \in P$ is a simplex there must be a unique edge for each pair of strategies belonging to the same S_j. Extending this argument up to the boundary faces we are able to guarantee the number of boundary faces is $\sum_{j=1}^{N} l_j$ where each face is labelled uniquely from the set S. In particular each pure strategy from S is used to label exactly one boundary face.

Example 4.14
Figure 4.15 is the covering seen in Figure 4.7, from Example 4.12, labelled as described above.

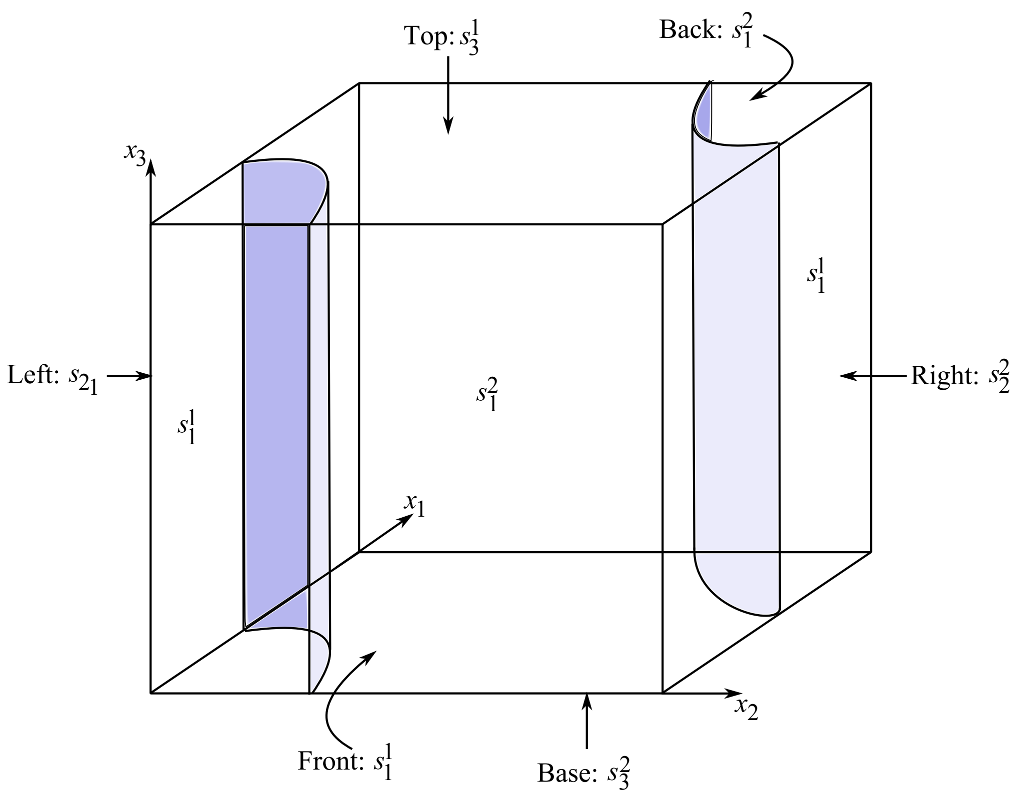


Figure 4.15: Labelled covering

(End Example)

Remark
The labelling described in this section satisfies the same conditions as that used by Shapley [Shapley, 1974].

To complete this section we describe how a labelled simplex is interpreted. Let Δ be a maximal simplex in η, where by maximal we mean Δ does not appear as a face of any other simplex and $\dim \Delta$ is maximal in η. Then Δ fully describes the intersection of a set of covering elements and boundary faces and so represents a situation from the game Γ^*. The labels assigned to the vertices of Δ provide us with the pure strategies chosen by all players in the mixed strategy situation $\mathbf{p}$ it represents. Additionally for every $\mathbf{p^{(i)}} \subset P^{(i)}$, $i = \{1, \ldots, N\}$, the vertices also identify player i's optimal strategies for situation $\mathbf{p^{(i)}}$. Let $W'_{\Delta,i}$ denote the set of vertices in Δ labelled by pure strategies from S_i associated to covering elements then these labels provide the optimal pure strategies for player i. The labels of the vertices associated to the boundary faces are contained in $\Sigma'_{\Delta,i}$ for all $i = \{1, \ldots, N\}$ and are the strategies **not** used by players $\{1, \ldots, N\}$. Observe these do not provide any information regarding optimality. Therefore for all $i = \{1, \ldots, N\}$ the mixed strategy situation played by player i uses pure strategies $S_i \setminus \Sigma'_{\Delta,i}$.

This completes the description of our representation of a generalised game Γ^*.

4.4 Properties of Nerves

Lemma 4.15 (Identifying Equilibrium Situations in η)
For all N-player generalised games Γ^*, every completely labelled simplex in the nerve η corresponds to at least one equilibrium situation. Further all equilibrium situations must be represented in this way.

Proof. Let $\mathbf{p} = (p_1, \ldots, p_N)$ be an equilibrium situation from Γ^*. The by definition, each player's set of pure best responses to situation $\mathbf{p^{(i)}}$ must be a superset of the support of p_i. Consequently, all pure strategies not belonging to the support of p_i must correspond to boundary faces of the original covering and so the set of labels assigned to vertices corresponding to covering elements and boundary faces must be disjoint. Finally by non-degeneracy the dimension of the simplex representing this situation in η must be $m' - 1$ and therefore a simplex representing an equilibrium situation is completely labelled.

Remark
Observe this is a consequence of property 2 of Definition 4.3 where for a given player the convex hull of all pure best responses must also be a best response.

Let Δ represent situation $\mathbf{p}$ in the nerve. Then let $W'_{\Delta,i}$ contain the vertices corresponding to the strategies belonging to the support of player i's optimal strategy for situation

$\mathbf{p^{(i)}}$, and $\Sigma'_{\Delta,i}$ contain those strategies which are not in the support for p_i. Then $W'_{\Delta,i}$ corresponds to the vertices representing covering elements in η and $\Sigma'_{\Delta,i}$ to those representing the boundary faces. Since by assumption $\mathbf{p}$ is an equilibrium situation, for all $i = \{1, \ldots, N\}$, $S_i \setminus \Sigma'_{\Delta,i} = W'_{\Delta,i}$, therefore $W'_{\Delta,i} \cup \Sigma'_{\Delta,i} = S_i$ and thus Δ is completely labelled or an equilibrium simplex.

For the converse, observe equilibrium simplices in Δ correspond to those situations where for all players $i = \{1, \ldots, N\}$ the optimal strategy for player i coincides with the strategy he has played. This is the definition of an equilibrium situation. Finally assume Δ is not completely labelled then, in particular, the support for the mixed strategy played by player i contains a pure strategy which is not optimal. Therefore player i's payoff can be improved by an alteration in his strategy alone and hence Δ does not correspond to an equilibrium situation. □

We return to the non-degeneracy condition of Γ^*. In Section 4.2 we discussed how the intersections of the coverings over $P^{(i)}$ must be smooth manifolds. This condition naturally extends to the coverings over P. In traditional non-degenerate games we insist intersections of these functions must be transverse. Therefore in the non-degenerate generalised game all intersections of covering elements labelled by the same S_i must intersect transversely with the intersection of covering elements labelled by the same S_j. (By definition the boundary intersections are automatically transverse).

Lemma 4.16 (Dimension of Nerve for Non-Degenerate Generalised Game)
In the non-degenerate game Γ^*, $\dim \eta = m' - 1$.

Proof. Given the condition of non-degeneracy ensures all intersections must be transverse we can immediately conclude in the non-degenerate game Γ^* there can be at most m' covering elements and boundary faces intersecting at any one point. We then observe such intersections can be identified at the vertices of the boundary and consequently the dimension of η is $m' - 1$ as required. □

Consider a 0-dimensional intersection in the covering of P lying away from ∂P. Then by Lemma 4.16 this corresponds to an intersection of m' covering elements. The conditions of non-degeneracy ensure the intersection of any subset of covering elements must be transverse and consequently each combination of $(m'-1)$ covering elements must intersect with dimension 1. Therefore a 0-dimension intersection in the covering is the end point of exactly m' distinct edges, each corresponding to a unique face in the simplicial complex, and locally η will satisfy the properties of non-ramification.

Remark

The term locally refers to nerve of the covering elements (and boundary faces) formed form the point of intersection and those edges leading to it only.

Remark

Compare our definition of non-degeneracy with that used by Shapley in [Shapley, 1974]. Then the discussion above has already included points 2 and 3 of Shapley's definition (given in Chapter 1.7). Property 1 is also contained within the conditions of transversal intersections. For if a covering element is not of dimension $(m' - 1)$ then any non-empty intersection with this covering elemet would contradict the dimension of a transversal intersection. Then since every covering element must be contained within the bounded space $P_1 \times \cdots \times P_N$, which is of dimension $m' - 1$, all covering elements must be of the required dimension.

Observe when the 0-dimensional intersection involves boundary faces there will be strictly less than m' edges which have the original intersection as an end point. This is because each edge leading to the intersection corresponds to a face in the nerve shared between two maximal simplices. Therefore if faces of a simplex belong to the boundary the number of edges the original intersection is an end point to must be reduced. However the nerve will remain locally non-ramified.

In light of this, and Definition 2.19, we deduce the nerve of any non-degenerate game will satisfy the following definition.

Definition 4.17 (Non-Degenerate Nerve)
The nerve η of a non-degenerate generalised game Γ^* satisfies the following three properties:

1. $\dim \eta = m' - 1$
2. *Locally each intersection in the covering corresponds to a simplex satisfying the conditions of non-ramification.*
3. *The process of regularisation over the entire nerve η results in a non-ramified simplicial complex.*

Relating back to the Shapley's interpretation of the Lemke-Howson paper, we comment on one final observation. In Chapter 3.6 we discussed how taking the dual or nerve of the simplicial complex $\mathcal{S}$ would result in an object satisfying Shapley's definition of $F_1 \times F_2$ (to the appropriate dimension). Constructing the nerve of the coverings over P will result in a simplicial complex, which we will show satisfies the definition of $\mathcal{S}$, and further the nerve of this complex will produce a labelled covering isomorphic to the original. Consequently the coverings defined over P can be associated to the graph $F_1 \times F_2$, i.e., nodes/edges in P (formed as a result of covering elements intersecting) correspond to nodes/edges in $F_1 \times F_2$. Of course this is all with dimension considerations in mind (See Chapter 3.6).

4.5 Proving Nash's Theorem

Our description of a non-cooperative generalised game Γ^* has been shown to be a simplicial complex. Therefore to prove Nash's Theorem for non-degenerate games we need to first show the nerve η is non-degenerate, and so in particular η satisfies Definition 4.17, and secondly that η is a particular example of the simplicial game complex $\mathcal{S}$ as given in Definition 3.1. Once this has been achieved we can apply Nash's Theorem for Simplicial Game Complexes, Theorem 3.11, to η to achieve our result.

We begin by proving η satisfies Definition 4.17 (we will se this will necessarily include a process to ensure η also satisfies the boundary conditions of $\mathcal{S}$). Properties 1 and 2 are automatic from the definition of transverse intersections. This just leaves property 3 and the use of regularisation to form a non-ramified simplicial complex. The following example illustrates a case where regularisation is required.

Example 4.18
Continuing from Example 4.12. From Figure 4.7 it is clear that at any point of intersection within P there can be at most two covering elements associated to any given player. We focus our attention to intersections involving the maximum of six covering elements. It is easy to construct examples where such covering elements have either 0,1 or 2 points of intersection. In this example we assume the latter case for illustration. Observe if all six covering elements have a non-empty intersection then by Lemma 4.15, such points must necessarily be equilibrium situations of Γ^*. Fix one pair of covering elements and consider the intersection of the coverings elements for the remaining two player's. The fixed covering elements will form a "cylinder", as seen in Figure 4.7. Therefore consider the intersection under investigation as the intersection of covering elements associated to 2 player's over a cylinder. Figure 4.16 shows the fixed cylinder and the two coverings for the remaining 2 player's.

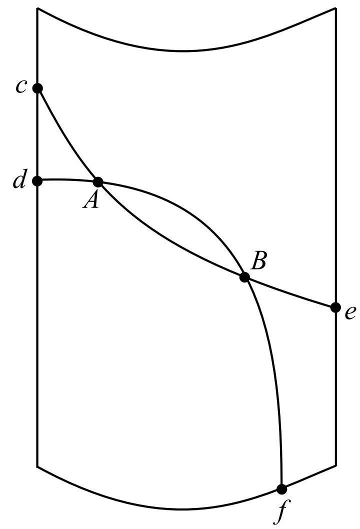


Figure 4.16: Intersection of 6 covering elements occuring at 2 distinct points

In Figure 4.16 the line joining c and e defines one covering while the line joining d and f defines the second covering. These coverings clearly intersect in 2 distinct places.

In the nerve η, intersections A and B are both represented by the same simplex Δ. As such within the nerve environment it will be impossible to distinguish between these two situations. Now consider the larger simplicial complex and let $\Delta_c, \Delta_d, \Delta_e$ and Δ_f represent the intersections labelled c, d, e, f respectively in Figure 4.16. Then each of these simplices share a face with Δ. However Δ_c and Δ_e share the same face of Δ and consequently the conditions of non-ramification are violated. The same is true for Δ_d and Δ_f.

Let Λ denote the shared face of Δ, Δ_c and Δ_e. To rectify the problem of non-ramification perform the regularisation operation, Definition 2.17, on simplices Δ_c and Δ_e. Then Δ_c and Δ_e become detached from Δ and Λ now satisfies the conditions of non-ramification. Repeating for Δ_d and Δ_f will result in Δ becoming disconnected from the nerve and as such we can remove it from the simplicial complex. Recall Δ corresponds to two equilibrium situations. Therefore, after the removal of Δ, the parity of equilibrium simplices in η will be equal to the parity of equilibrium situations in Γ. In terms of the covering, this process has the affect of perturbing the covering elements into a covering belonging to the family shown in Figure 4.17.

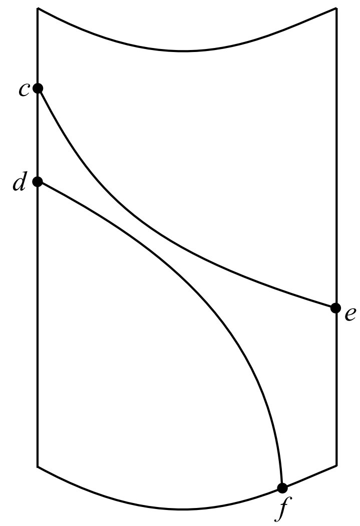


Figure 4.17: Empty intersetion of 6 covering elements

It is clear the coverings shown in Figure 4.17 is an example where the six covering elements have an empty intersection.

(End Example)

We return to the nerve η and to examining the form of this simplicial complex. Recall in the non-degenerate generalised game $\dim \eta = m' - 1$. Consider an intersection of m' covering elements and boundary faces and let this situation be represented by Δ in the nerve. First consider the scenario when this intersection is a single point, and in particular is connected. Then since Γ^* is non-degenerate each face of Δ must represent at most one 1-dimensional edge in the covering and consequently each face belongs to at most one other simplex. Therefore at such points the simplicial complex is non-ramified and no work needs to be done.

Now consider the case when the same set of covering elements and boundary faces intersect in multiple disjoint places, as seen in Example 4.18, then each connected component of the intersection will be represented by the same simplex Δ. In particular assume a set of m' covering elements and boundary faces have a non-empty intersection occurring in k isolated points, $T_j,\ j \in \{1, \dots, k\}$. Then locally, by non-degeneracy, each intersection must behave as in the simple case described in the previous paragraph when $k = 1$. Therefore, since Γ^* is a non-degenerate game, leading from each vertex must be at most m' distinct edges, each one corresponding to a maximal face of Δ.

In the covering, all of the edges formed from the intersection of $(m'-1)$ of the m' covering elements and boundary faces under consideration can be divided into two sets: internal and external. An internal edge connects 2 of the vertices from the set of k isolated intersections, the rest are external. For all $i, j \in \{1, \ldots, k\}$ and for each intersection T_j we define the following k sets:

$$E_j \ \cup \ \{E_{ji} \,|\, i \in \{1, \ldots, i-1, i+1, \ldots, N\}\} \tag{4.15}$$

where E_j contain those edges with T_j as an end point and an empty intersection with all other T_i, and the set E_{ij} contains those edges which have vertices T_i and T_j as an end points. The union of these sets form the set of all edges leading from T_j such that no edge appears in multiple sets. Observe the union $E := \bigcup_{j=1}^{k} E_j$ contains all external edges.

If external edge e_1 belongs to E_j for all $j = \{1, \ldots, k\}$ then the number of times this edge is classified as external (and hence belongs to the set E) is equal to the number of vertices k. Without loss of generality assume $e_1 \in E_{12}$ then we must have $e_1 \in E_{21}$ and by the properties of non-degeneracy $e_1 \notin E_1$ and $e_1 \notin E_2$. Assume in addition $e_1 \notin E_3$. Then $e_1 \in E_{3j}$ for some $j \in \{1, 2, 4, \ldots, N\}$. However since we know $e_1 \in E_{12}$ and $e_1 \in E_{21}$ by degeneracy conditions we also have $j \neq \{1, 2\}$. Therefore for some $l \in \{4, \ldots, N\}$ we have $e_1 \in E_{3l}$ and therefore $e_1 \notin E_l$ either. Consequently the parity of the occurrence of a given edge as an external edge coincides with the parity of k. In particular all external edges occur with the same parity.

Assume e_1 belongs to E_j for M_1 intersections T_j where $1 \leq M_1 \leq k$. Then e_1 corresponds to the face Λ_1 of Δ which also belongs to M_1 additional (maximal) simplices in η. Label these additional simplices by $\Delta_1, \ldots, \Delta_{M_1}$ and assume $M_1 \geq 2$ as then Λ_1 fails the conditions of non-ramification. While possible arbitrarily pair simplices $\Delta_1, \ldots, \Delta_{M_1}$ and perform the detachment operation with respect to Λ_1. If M_1 is even, after regularisation Δ no longer shares the face Λ_1 with a second maximal simplex in η and thus becomes detached. If M_1 is odd then Δ shares Λ_1 with just one other maximal simplex and thus Δ remains connected in the simplicial complex and the conditions of non-ramification are met. Repeat for all external edges. Since for all external edges $e_i \in E$, M_i is of the same parity as M_1, Δ is either completely detached from the nerve, in which case we can remove it, or all faces of Δ are non-ramified. However the parity of M_i is also equal to the parity of k and we observe either all k vertices are equilibrium situations or none of them are. Therefore removing Δ in the case k is even is equivalent to excluding an even number of equilibrium situations from η. When k is odd regularisation removes all but 1 equilibrium simplex or an even number of equilibrium situations. In either case the resulting simplicial complex η is a non-ramified simplicial complex of dimension $(m'-1)$ such that the parity of equilibrium simplices in η is equal to the parity of equilibrium situations in Γ^*.

Remark

By construction of the nerve, any edge not belonging to E_j, for any $j \in \{1, \ldots, N\}$, corresponds to a face belonging to Δ only.

Remark

For $j = \{1, \ldots, k\}$ let e_j be an edge with end point T_j such that all e_j are formed from the same set of covering elements and boundary faces. Then if k is odd at least one $e_j \in E_j$ is external. In particular each face of Δ is shared with at least one other maximal simplex in η. This is not the case when k is even and this introduces the complication of additional boundary faces. However in this case regularisation detaches Δ from η and as such, problems surrounding the boundary are removed.

The process described above assumes if the face Λ belongs to simplices $\Delta, \Delta_1, \ldots \Delta_l$, where $l \geq 2$ so the property of non-ramification fails, then $\Delta \neq \Delta_1 \neq \ldots \neq \Delta_l$. In particular each edge which corresponds to the face Λ of Δ has a second distinct end point. We now consider when this condition does not hold. To do this we need the following definition.

Definition 4.19 (Multiple Edge)

Call e a multiple edge *if there is at least one other edge $e' \neq e$ which corresponds to the same face Λ of Δ and Δ'. Call the number of such edges the* multiplicity *of Δ' and of e.*

Remark

If in the nerve e represents the face Λ shared by simplices Δ, Δ_1 then e is said to have multiplicity 1 if a second edge e' which also corresponds to the same face Λ of simplices Δ and Δ_1 cannot be identified.

Definition 4.20 (Single Edge)

If an edge e is not a multiple edge then call e a single edge*.*

For the set $T_1, \ldots, T_k$ of isolated points, let e and e' both be external edges represented by the face Λ in η. Then if for both e and e', Λ corresponds to the same simplex Δ' the edge e (and consequently e') has multiplicity 2 (provided no other edge e'' also satisfies this condition, in which case the multiplicity will increase). In particular e' is the result of the intersection of the same $(m'-1)$ covering elements and boundary faces as e and $e \cup e'$

forms a disconnected edge within the covering. It follows that if e is a multiple edge there is another set of isolated points $T'_1, \dots, T'_{k'}$ for which e (and e') is a multiple external edge.

Suppose edge e has multiplicity $h \geq 2$. Then within the covering there are h edges leading to h different end points. However within the nerve these points are represented by a single face and a single simplex and in particular the property of non-ramification holds. When h is even this can interfere with the outcome of the regularisation process described previously. In the case when all edges are single edges (i.e. all external edges have multiplicity 1), when k is even we showed regularisation removed the simplex Δ from the nerve and when k is odd Δ remained and satisfied the conditions of non-ramification. If this was not the case then either the parity of equilibrium simplices is not equal to the parity of equilibrium situations in the game or we introduce boundary faces in η which do not satisfy Definition 3.1. Therefore we need to produce the same result when the covering contains multiple edges. Assume there is a face Λ of Δ which corresponds to a simplex Δ_1 more than once, in particular we can identify at least one multiple edge. Then while the pair Δ and Δ_1 satisfy the conditions of non-ramification (considered in isolation from the rest of the simplicial complex) they represent multiple situations in the game.

For each face Λ of the simplex Δ there are four scenarios which can arise and these are presented now along with the action required to ensure the the nerve not only satisfies the definition of non-degeneracy but also the boundary requirements of $\mathcal{S}$. Recall k denotes the maximum number of isolated points represented by Δ and we are assuming there is at least one multiple edge. Additionally, from above, we know the parity of external edges (counted with multiplicities) is equal to the parity of k and if e is a multiple edge. We do not assume the multiplicity of each multiple edge leading from Δ is of the same parity.

- **Case 1: k is even and Λ corresponds to an even number of single edges**

  In η the face Λ corresponds to an even number of edges in the covering. Since an even number of these are single edges it must be that an even number have a multiplicity greater than or equal to 2. Further we can deduce there must be an even number (or zero) of multiple edges with odd multiplicity. If simplex Δ_1 is of odd multiplicity then in the covering Δ_1 represents an odd number of (possibly equilibrium) situations from the game. Equivalently in the nerve representation an even number of such situations are ignored thus the parity remains unchanged. This is not the case when the multiplicity of Δ_1 is even. In this case an odd number of situations are excluded and we cannot compare the parity of equilibrium simplices in η to equilibrium situations from Γ^*. Since it is only the simplices of even multiplicity which cause problems, identify all simplices of odd multiplicity and those which can be identified as the end points of a single edge (**single simplices**) into a single set. In this case this set will contain an even number of simplices. Arbitrarily pair these simplices together, then perform the detachment operation as described previously with respect to the shared face Λ and simplex Δ. After this process has been completed Δ only shares a face with simplices of even multiplicity.

When k is even we saw previously this may introduce boundary faces in η which fail Definition 3.1 therefore our aim is to remove Δ during regularisation. To achieve this detach every single simplex Δ' of even multiplicity from Λ. Then regularisation (Definition 2.17) will remove Δ from the nerve, thus removing an even number of equilibrium situations and any unwanted 'boundary faces' introduced during detachment.

- **Case 2: k is even and Λ corresponds to an odd number of single edges**

  Once again the face Λ corresponds to an even number of edges in the covering. This time an odd number of these are single edges and so an odd number must be multiple edges. In particular there must be an odd number of edges with odd multiplicity. Then the set of single simplices and simplices of odd multiplicity contains an even number of elements. Repeat as in Case 1 to achieve the same result.

- **Case 3: k is odd and Λ corresponds to an even number of single edges**

  In this case Λ now represents an odd number of edges of which an odd number must be multiple edges of odd multiplicity. Now the set of single simplices and simplices of odd multiplicity contains an odd number of elements. While possible pair simplices from this set together and detach with respect to face Λ. Then Δ shares the face Λ with one simplex from this set and regularisation does not remove Δ from the nerve as seen in Cases 1 and 2. Finally detach all simplices which occur with an even multiplicity from the face Λ to prevent problems with non-ramification. Then Δ satisfies the conditions of non-ramification and the difference between equilibrium situations and simplices is an even number.

- **Case 4: k is odd and Λ corresponds to an odd number of non-multiple edges**

  In this final case Λ once again corresponds to an odd number of edges but this time there must be an even number of edges with odd multiplicity so once again the set of single simplices and simplices of odd multiplicity contains an odd number of elements. Repeat as in Case 3 to achieve the same result.

We now reach our final consideration. So far we have considered non-ramification from the view point of a fixed simplex Δ, where Δ may have a face Λ which belongs to 2 or more additional simplices. In the covering this is equivalent to a disconnected edge e where at least one end point of every connected component corresponds to the same simplex Δ in η. We now consider the scenario where a $(m'-2)$-face Λ belongs to more than 2 distinct simplices but where the end points of the connected components of the corresponding edge e in the covering need not be represented by the same simplex.

For ease of representation we consider each connected component of e as a separate edge. Then these edges can be classified as either **closed** or **open**. A closed edge will have 2 distinct end points which will correspond to two distinct simplices Δ_1, Δ_2 in the nerve with shared face Λ. An open edge will have just 1 end point and so (with respect to this one edge) Λ belongs to just one simplex and thus is a boundary face.

Suppose Λ is the face of simplices $\Delta_1, \ldots, \Delta_l$. Then by the previous cases we can assume no simplex represents more than one situation in the game and thus they must all be distinct. Observe that the definition of external and internal no longer applies in this setting as the points of non-ramification to which these are associated have now been removed. We partition the l simplices into two sets, depending on if they correspond to an end point of an edge which is closed or open. Let the first set contain the l_1 simplices corresponding to closed edges, then these simplices must necessarily occur in pairs. The second set consists of the remaining l_2 simplices corresponding to the open edges. Note it must be that $l = l_1 + l_2$. For every pair of $(m'-1)$-simplices in the first set perform the detachment operation with respect to Λ. Follow this by the detachment of every individual $(m'-1)$-simplex in the second set with respect to Λ. Then Λ is detached from the nerve and so is removed during regularisation. Repeat for all faces Λ which satisfy this property; this will necessarily be the remaining faces in η which fail the conditions of non-ramification.

Finally we show our regularised simplicial complex η satisfies Definition 3.1. The first step to achieve this is to describe Γ^* in the same terminology as $\mathcal{S}$. Let $\mathbf{m}' = \{1, \ldots, m'\}$ where $m' = \sum_{i=1}^{N} l_i$ and define ϕ' to be a bijective **labelling function** from $S_1 \times \cdots \times S_N$ to $\mathbf{m}'$. Then there is a natural partition of $\mathbf{m}'$ into sets $\mathbf{m}'_\mathbf{i} = \{m_{i-1}+1, \ldots, m_i\}$ where for all $i = \{1, \ldots, N\}$, $\phi(S_i) = \mathbf{m}_\mathbf{i}$. Therefore we have $|S_i| = l_i = m_i - m_{i-1}$. Label the vertices of η from the set $\mathbf{m}'$ according to the following rules. A vertex associated to covering element $B_i(s)$ is labelled by $\phi(s)$. If a vertex is associated to the boundary face which contains all mixed strategy situations not involving $s' \in S := S_1 \times \cdots \times \cdots S_N$ then label the vertex by $\phi(s')$ (this choice will be unique, no two boundary faces will have the same label and all labels from the set S will be used exactly once). Let Δ be a maximal simplex then Δ represents a situation form Γ^*. Let $W'_{\Delta,i}$ be the subset of vertices from $v(\Delta)$ associated to covering elements with label from $\mathbf{m}'_\mathbf{i}$ and $\Sigma'_{\Delta,i}$ be the subset of vertices from $v(\Delta)$ associated to boundary faces with labels from $\mathbf{m}'_\mathbf{i}$. This is the direct analogy of $W_{\Delta,i}$ and $\Sigma_{\Delta,i}$ from Definition 3.1. Then for all players $i = \{1, \dots, N\}$ those vertices in $\Sigma'_{\Delta,i}$ define the situation Δ represents in P by identifying the pure strategies not used by player i, or equivalently selected with probability 0 in their mixed strategy. The sets $W'_{\Delta,i}$ then identify player i's optimal pure strategies for the situation represented by Δ. Recall for all $i = \{1, \dots, n\}$, P_i is a simplex of dimension $m_i - m_{i-1} - 1$, therefore inline with Definition 3.1 the simplex P_i is equivalent to the simplex Σ_{m_i}.

Properties 1 and 2 of Definition 3.1 are confirmed by a simple observation. For all $i = \{1, \dots, N\}$, the sets $\Sigma'_{\Delta,i}$ contain vertices from the simplex $P_i \subset P$. Recall the nerve of P is equal to $\partial P_1 * \dots * \partial P_N$. Then the restriction of η to vertices from $v(P_1) \cup \dots \cup v(P_N)$ is equal to the union of $(\Sigma'_{\Delta,1} \cup \dots \cup \Sigma'_{\Delta,N})$ for all maximal simplices Δ, this by definition of the covering must be equal to the nerve of P. For the second property it is clear $W_{\Delta,i} \neq \emptyset$ for all $i = \{1, \dots, N\}$.

Observe property 4 of Definition 3.1 contains two parts. The first is the verification η is a non-ramified simplicial complex, which we have just shown, and second the boundary of η coincides with the boundary of $\mathcal{S}$.

Lemma 4.21
For a non-degenerate game Γ^*, the boundary of η satisfies the definition of $\partial\mathcal{S}$.

Proof. Let Δ be a maximal simplex in $\mathcal{S}$, then the face $\Lambda \subset \Delta$ belongs to the boundary of η iff that face does not belong to any other simplex. Equivalently the set of covering elements and boundary faces which form Λ all have a non-empty intersection with exactly one other covering element or boundary face. It is clear to see, the properties of non-ramification (which can be assumed by the above) only ensure this can happen when all vertices associated to a player i represent boundary faces of P_i. Equivalently, in the corresponding face, no vertex with a label from S_i is associated to a covering element. Therefore Δ has a face on the boundary exactly when $W'_{\Delta,i}$ contains one element. □

Finally, property 3 is equivalent to determining Lemma 3.8 holds for the non-degenerate simplicial game complexes; as it property 3 which ensures this result and consequently the proof of Nash's Theorem for Simplicial Game Complexes, Theorem 3.11.

For clarity we introduce some new notation. Let Γ^*_N be the N-player non-cooperative generalised game where each player's optimal strategies are determined using a total order $\succ_i$ over P. Now define Γ^*_{N-1} to be the restriction of Γ^*_N to an $N-1$-player game. In particular assume player N fixes his strategy choice to $s_N^{l_N} \in S_N$ and the remaining players determine their optimal strategies once again using the total order $\succ_i$ but this time over the set $P^{(N)} := P_1 \times \cdots \times \cdots \times P_{N-1} \times \{s_N^{l_N}\}$. Then observe $P^{(N)}$ is a face of the product space P and Γ^*_{N-1} is the game described by the $N-1$ coverings over the face $P^{(N)} \subset P$.

Let η_N be the nerve formed from the game Γ^*_N then we have already shown that this can be compared directly to $\mathcal{S}_N = \mathcal{S}$ (with the exception of property 3). Then if η' is the nerve for the sub-game Γ^*_{N-1} the same regularisation procedure can be carried out to demonstrate it satisfies the definition of $\mathcal{S}_{N-1} = \mathcal{S}'$.

Suppose η_N is regular (Definition 2.20) then we can attain the subcomplex η' representing the $(N-1)$-player game in the same manner as described in Definition 3.6. However since the co-dimension of η_N and η' is at least 2 we cannot determine if η' is also a non-ramified complex. If η' is regular then we are done. Alternatively we perform the detachment operation on η' as described above for the case when η_N is not regular. Therefore for this case Lemma 3.8 applies and property 3 of Definition 3.1 is satisfied.

Alternatively η_N needs to be regularised before it satisfies the conditions of non-ramification. However due to the nature of the regularisation operation there is no clear way of arriving at η' from η_N (unlike the case when η_N is regular). This has the affect of being unable to use Lemma 3.8 so we provide Lemma 4.23 in its place. Assume the notations η' and η_N refer to non-ramified simplicial complexes and therefore may have been subjected to regularisation.

Definition 4.22 (Sub-equilibrium Situation)
Define a sub-equilibrium situation *to be one where all conditions of an equilibrium situation are satisfied except perhaps with regards to pure strategy $s_N^{l_N} \in S_N$. In terms of the N coverings over P a sub-equilibrium situation is represented by the intersection of $m'-1$ covering elements and boundary faces labelled by $\{1, \ldots, m_n - 1\}$.*

Lemma 4.23
If there are an odd number of equilibrium simplices in η' then there are an odd number of equilibrium simplices in η_N.

Proof. Assume there are an odd number of equilibrium simplices in η' then by Lemma 4.15 (and regularisation discussions) there are an odd number of equilibrium situations in the game Γ^*_{N-1}.

Recall Γ^*_{N-1} is represented by a covering which appears across a face belonging to the boundary of P. In particular within the join $P^{(N)} * \sigma$, where $\sigma \subset P_N$ and $v(\sigma) = \{m_{n-1} +$

$1, \ldots, m_n - 1\}$, we can identify an odd number of sub-equilibrium situations in the $(m' - 2)$-faces of P. Observe this process must define all sub-equilibrium situations belonging to the boundary of P.

By definition, sub-equilibrium situations belonging to the boundary of P are represented as sub-equilibrium faces lying in the boundary of η_N. For η_N create the graph G as defined in Definition 3.10, then the number of extreme vertices in G is equal to the number of sub-equilibrium faces lying in $\partial\eta_N$. Therefore G contains an odd number of extreme vertices and by Lemma 2.22 Γ^*_N must have an odd number of equilibrium situations. □

Consequently when the simplicial complex is a nerve of a covering defined via a total order and is not regular an equivalent form of Lemma 3.8 has been proved. In particular for simplicial complexes satisfying this property Lemma 4.23 should be used in place of Lemma 3.8 in the proof of Theorem 3.11, Nash's Theorem for Simplicial Complexes.

We now prove Nash's Theorem for non-degenerate generalised games.

Theorem 4.24 (Generalised Nash's Theorem for Non-Degenerate Games)
There are a finite and odd number of equilibrium situations in a non-degenerate generalised game Γ^*.

Proof. For a non-degenerate, generalised game Γ^* we know the associated nerve η is finite and so the number of equilibrium situations it contains must be finite too. Additionally we know η a non-ramified simplicial complex without changing the parity of equilibrium situations it represents. In particular, in this form η is a simplicial complex satisfying the definition to be a non-degenerate simplicial game complex $\mathcal{S}$ of the order n as defined in Definition 3.1. If η is regular then use Theorem 3.11 directly to show η contains a finite and odd number of equilibrium simplices.

Alternatively if η needs to be regularised then Nash's Theorem for Simplicial Game Complexes, as proved for simplicial complexes defined as nerves using Lemma 4.23, should be used. Then the result is once again achieved directly and η contains a finite and odd number of equilibrium simplices.

In both cases, Lemma 4.15 can then be used to demonstrate Γ^* contains a finite and odd number of equilibrium situations.

$\square$

Remark
Restricting ourselves to games which generate regular nerves, and hence do not require Lemma 4.23, is an interesting and rich area of game theory. However we will not pursue this interesting area in this Thesis.

4.6 Nash's Theorem for Traditional Non-Cooperative Games

We observed earlier in this Chapter any result for generalised games defined via a total order would automatically hold for traditional games defined by polylinear payoff functions. In particular Theorem 4.24 proves Nash's Theorem, as given in Theorem 1.13, for non-degenerate non-cooperative traditional games. However generalised games may require Lemma 4.23 to complete the proof. We now show this Lemma is not necessary in the traditional case.

Let Γ_N represent an N-player non-degenerate, non-cooperative traditional game where each player's payoff function $G_i : P \mapsto \mathbf{R}$ is polylinear. Then Γ_{N-1} is the $N-1$-player subgame of Γ_N defined when player N fixes his strategy choice to $s_N^{l_N} \in S_N$ and the payoff functions for the remaining $N-1$ player's $i = \{1 \ldots, N-1\}$ are defined by $G_i : P^{(N)} \times \{s_N^{l_N}\} \mapsto \mathbf{R}$. Observe Γ_{N-1} is contained within $P_{N-1} := P_1 \times \cdots \times P_{N-1} \times \{s_N^{l_N}\}$ which is a face of the product space P. In particular Γ_{N-1} is the game described by the $N-1$ coverings over the face $P^{(N)} \subset P$.

Let η_N be the nerve of the game Γ_N then we have already shown this can be compared directly to $\mathcal{S}_N = \mathcal{S}$. Equivalently if η' is the nerve of the sub-game Γ_{N-1} then regularisation can once again be applied to ensure it satisfies the definition of $\mathcal{S}_{N-1} = \mathcal{S}'$. The nerve η_N will either be regular or regularisation will be required in order for it to be a non-ramified simplicial complex. In the first case regularisation is not required and we repeat the argument from the previous section. In particular it must be that η' can be constructed from η_N in the same manner as described in Definition 3.6. If necessary regularise η' then Lemma 3.8 applies and property 3 of Definition 3.1 is satisfied.

Alternatively η_N is regularised to form a non-degenerate simplicial complex. Then in this case there is no clear way of arriving at η' from η_N. This has the affect of being unable to use Lemma 3.8. To overcome this, without using Lemma 4.23, we make use of a famous result by Bubelis [Bubelis, 1979] which states any N-player non-degenerate, non-cooperative game can be reduced to a 3-player game such that the equilibria in one corresponds to the equilibria in the other. This is also described in Chapter 2, Section 6 of [Vorob'ev, 1994].

By Bubelis' Theorem, Γ_N can be translated into a 3-player game, Γ_3, such that the set of equilibrium situations in Γ_N has a direct correspondence to the equilibria points in Γ_3. Equivalently if η_3 represents the nerve of Γ_3 then the number of equilibrium situations represented in η_3 is equal to the number in η_N. Observe the subgame associated to η_3 is a bimatrix game Γ_2 with nerve η_2 where the codimension of η_2 in η_3 is at least 2. Therefore the regularisation procedure applied to η_3 (if required) does not affect η_2. Then by Theorem 5.1 (stated and proved in the following chapter, Chapter 5) η_2 is regular. As such η_2 satisfies the definition of $\mathcal{S}'$, regardless of the need for η_3 to be regularised. Therefore once again Lemma 3.8 applies and property 3 of Definition 3.1 is satisfied. In particular the proof of Theorem 4.24 can use Theorem 3.11 as originally poved using Lemma 3.8.

Chapter 5

Bimatrix Games

In Chapter 3 we defined the simplicial game complex $\mathcal{S}$ and in Chapter 4 showed its description contained the simplicial complexes generated from our definition of generalised games. Observe the number of players in a game Γ^* is equal to the order of the corresponding simplicial complex $\mathcal{S}$. In this Chapter we first show the nerve of traditional bimatrix games Γ are non-ramified simplicial complexes without the need for regularisation. We then suggest an alternative, and simpler, formalisation of the simplicial game complex $\mathcal{S}$ for this important subclass of games. For this new simplicial complex we replicate the proofs of the previous two chapters and show our simplification is a specific form of the general case.

5.1 Nerves of Bimatrix Games

Lemma 5.1
For every non-degenerate bimatrix game Γ, the nerve of the coverings generated by the polylinear payoff functions G_1 and G_2 is a regular non-ramified simplicial complex of the appropriate dimension.

Proof. In Γ observe all covering elements defined by G_1 and G_2 are convex polyhedra. In particular for $i = \{1, 2\}$ the covering with respect to player i over P consists of a maximum of l_i connected covering elements where each pure strategy $s_i^j \in S_i$ describes at most one set $B_i(s_i^j)$. Therefore the intersection between any pair of covering elements (and boundary faces) must be a connected subset of P and as such η is a non-ramified simplicial complex. The conditions of non-degeneracy ensures η is of the correct dimension and the boundary conditions automatically follow through from Lemma 4.21. Consequently the nerve η is a regular simplicial complex of appropriate dimension as required.

$\square$

5.2 An Alternative Simplicial Game Complex

Let $\mathcal{S}^*$ denote the new formalisation of the simplicial game complex $\mathcal{S}$. For a finite natural number m the simplicial game complex $\mathcal{S}^*$ will rely on the partitioning of the set $\mathbf{m} := \{1, \ldots, m\}$ into two elements. For $i = \{1, 2\}$ each partition element is defined by $\mathbf{m_i} = \{m_{i-1}+1, \ldots, m_i\}$ with $m_0 = 0$, $m_2 = m$ and $|\mathbf{m_i}| > 1$ such that $\mathbf{m} = \mathbf{m_1} \cup \mathbf{m_2}$ but $\mathbf{m_1} \cap \mathbf{m_2} = \emptyset$. Once again let V denote the set of vertices of $\mathcal{S}^*$ where V satisfies the condition $|V| \geq m+2$. We define $\phi : V \mapsto \mathbf{m}$ to be a surjective **labelling function** which assigns each vertex in $\mathcal{S}^*$ a unique label from a set of m possibilities. We can then identify and fix a subset $V' \subset V$ such that the restriction of ϕ to V' is bijective. Within the set $V \setminus V'$ we insist on being able to define two non-empty subsets, W_1 and W_2, such that $\phi(W_i) \subset \mathbf{m_i}$. Denote by Σ_{m_i} the simplex with vertices in V' such that $\phi(v(\Sigma_{m_i})) = \mathbf{m_i}$ then the dimension of Σ_{m_i} is $|\mathbf{m_i}| - 1 = (m_i - m_{i-1} - 1)$. We now use these preliminary notations to define our finite combinatorial simplicial game complex $\mathcal{S}^*$ with set of vertices V.

Definition 5.2 (Simplicial Game Complex $\mathcal{S}^*$)
For $i, j = \{1, 2\}$ and $i \neq j$, define $\mathcal{S}_{1i}$ to be $\partial\Sigma_{m_j}$. Then $\mathcal{S}_{2i}$ is an arbitrary complex with set of vertices $(v(\Sigma_{m_j}) \cup W_i)$ such that

(i) The restriction of $\mathcal{S}_{2i}$ on V' coincides with $\mathcal{S}_{1i}$

(ii) After the removal of some (possible zero) simplices from $\mathcal{S}_{2i} \setminus \mathcal{S}_{1i}$, the complex $\mathcal{S}_{2i}$ becomes an $(m-m_i+m_{i-1}-n+1) = (m-m_i+m_{i-1}-1)$ dimensional pseudomanifold with boundary $\mathcal{S}_{1i}$.

The simplicial game complex $\mathcal{S}^*$ is then the join defined by $\mathcal{S}^* := \mathcal{S}_{21} * \mathcal{S}_{22}$. If $\mathcal{S}_{21}$ and $\mathcal{S}_{22}$ are already simplicial pseudo-manifolds, the simplicial game complex $\mathcal{S}^*$ is called non-degenerate*.*

Remark
The process of the removal of simplices in (ii) of Definition 5.2 is not unique and may result in different pseudomanifolds for different processes.

Remark
The condition to be a pseudomanifold with boundary can be replaced by a weaker condition to be a *non-ramified complex with boundary* i.e. satisfying axiom 2 of Definition 2.7.

The definition of **equilibrium simplices** within the simplicial game complex $\mathcal{S}^*$ is identical to that given in Definition 3.3.

We now formulate and prove Nash's Theorem for our simplicial game complex $\mathcal{S}^*$.

Theorem 5.3 (Nash's Theorem for Non-Degenerate Simplicial Game Complex $\mathcal{S}^*$)
Every non-degenerate simplicial game complex $\mathcal{S}^*$ contains a finite and odd number of equilibrium simplices.

Proof. We begin by identifying a $(m-1)$-sub-equilibrium simplex lying in $\mathcal{S}^*$.

Let γ_2 be the $(m-m_1-2)$-face of $\mathcal{S}_{11}(=\partial\Sigma_{m_2})$ such that $\phi(v(\gamma_2)) = \{m_1+1,\ldots,m-1\}$. Recall $\mathcal{S}_{11}$ is the boundary of $\mathcal{S}_{21}$ then since γ_2 is a (maximal) simplex of $\mathcal{S}_{11}$ it must be that γ_2 lies in the boundary of $\mathcal{S}_{21}$. By definition, $\mathcal{S}_{21}$ is a non-ramified complex of dimension $(m-m_1+m_0-1) = m-m_1-1$ and therefore we are able to identify a unique vertex x from the set W_1 such that the simplex $\gamma_2 * \{x\}$ lies in $\mathcal{S}_{21}$. We assume without loss of generality $\phi(x) = \ell \in \mathbf{m_1}$.

Within $\mathcal{S}_{12}$ there is a face γ_1 such that $\phi(v(\gamma_1)) = \{1,\ldots,\ell-1,\ell+1,\ldots,m_1\}$. Since $\mathcal{S}^* = \mathcal{S}_{21} * \mathcal{S}_{22}$, the simplex $\gamma_1 * \gamma_2 * \{x\}$ is a $(m-2)$-dimensional sub-equilibrium simplex contained in $\mathcal{S}^*$.

By definition the dimension of $\mathcal{S}^*$ is given by

$$
\begin{aligned}
\dim \mathcal{S}^* &= \dim(\mathcal{S}_{21} * \mathcal{S}_{22}) \\
&= \dim(\mathcal{S}_{21}) + \dim(\mathcal{S}_{22}) + 1 \\
&= (m - m_1 + m_0 - 1) + (m - m_2 + m_1 - 1) + 1 \\
&= m - 1 \qquad\qquad (\text{since } \ m_0 = 0 \ \text{ and } \ m_2 = m)
\end{aligned} \tag{5.1}
$$

and since $\mathcal{S}_{22}$ also satisfies the properties of non-ramification, we are able to identify a unique vertex $y \in W_2$ such that $\gamma_1 * \gamma_2 * \{x, y\}$ is a $(m-1)$ simplex in $\mathcal{S}^*$. It is clear $\phi(v(\gamma_1 * \gamma_2 * \{x, y\})) \supseteq \{1, \ldots, m-1\}$ and therefore this simplex is sub-equilibrium.

Construct the undirected graph $G = (V, E)$, as defined in Definition 3.10 then, by the above discussion, $V \neq \emptyset$. In line with Lemma 2.22 describe a vertex $v \in V$ as **normal** if the simplex it represents in $\mathcal{S}^*$ is equilibrium. Then the property of non-ramification ensures the degree of a normal vertex can be at most one and that of a non-normal vertex at most 2. Such vertices are **extreme** if they are of degree 0 and 1 respectively. We complete the proof by determining there are an odd number of extreme vertices in G. On completion of this Lemma 2.22 can be used to deduce there are an odd number of normal vertices and hence equilibrium simplices in $\mathcal{S}^*$. We refer to the vertices in V by the simplices they represent in $\mathcal{S}^*$.

We claim $\gamma_1 * \gamma_2 * \{x, y\}$ is the only extreme vertex in G.

We first prove $\gamma_1 * \gamma_2 * \{x, y\}$ is an extreme vertex in G. This requires us to distinguish between the equilibrium and sub-equilibrium cases. Recall $\gamma_1 * \gamma_2 * \{x, y\}$ is sub-equilibrium and it is only the label assigned to y which is unknown. However we do know y is an element of W_2 and so in particular we must have $\phi(y) \in \mathbf{m_2} = \{m_1 + 1, \ldots, m\}$. This leaves us with two cases to consider, the first when $\phi(y) = m$ and the second when $\phi(y) \neq m$.

In case 1 we assume $\phi(y) = m$ and so $\gamma_1 * \gamma_2 * \{x, y\}$ is an equilibrium simplex. It is therefore a normal vertex in G and is extreme if and only if it is of degree 0. By definition, equilibrium simplices contain just one sub-equilibrium face, which in this case is given by $\gamma_1 * \gamma_2 * \{x\}$. Since we know γ_1 belongs to the boundary of $\mathcal{S}_{22}$ and $\gamma_2 * \{x\}$ belongs to $\mathcal{S}_{21}$, by recalling $\mathcal{S}^* = \mathcal{S}_{21} * \mathcal{S}_{22}$, Lemma 2.10 determines $\gamma_1 * \gamma_2 * \{x\}$ is a boundary face of $\mathcal{S}^*$. Therefore $\gamma_1 * \gamma_2 * \{x, y\}$ does not share its only sub-equilibrium face with another simplex in $\mathcal{S}^*$ and consequently the corresponding vertex in G is of degree 0 and is extreme.

For case 2 assume $\phi(y) \neq m$ then $\gamma_1 * \gamma_2 * \{x, y\}$ is a sub-equilibrium simplex. For $\gamma_1 * \gamma_2 * \{x, y\}$ to be extreme we show this is a vertex of degree 1 in G. Within this simplex there are two vertices with the same label. One of these vertices is necessarily y and let the second be denoted by z then $\phi(y) = \phi(z)$. Removing each vertex in turn will identify two sub-equilibrium faces. The first will be $\gamma_1 * \gamma_2 * \{x\}$ and is achieved by removing vertex $\{y\}$. Repeating the argument used in case 1 will once again determine this is a boundary face and consequently will not contribute to the degree of the vertex. To construct the second sub-equilibrium face we remove vertex z. We express this face as the join of $C^{(1)}$ and $C^{(2)}$ where $C^{(i)}$ is a simplex lying in $\mathcal{S}_{2i}$. Observe since $\phi(x) \in \mathbf{m_1}$ and $\phi(z) \in \mathbf{m_2}$ we cannot have the scenario $x = z$ and so the 1-simplex $\{x, z\}$ must belong to the sub-equilibrium face under consideration. In particular within each $C^{(i)}$ we can identify a vertex which does not belong to the boundary. We use Lemma 2.10 once again, but this time we determine the sub-equilibrium face does not lie in the boundary of $\mathcal{S}^*$. Consequently it must also be a sub-equilibrium face of another (sub)-equilibrium simplex in $\mathcal{S}^*$ and the corresponding vertex in G is of degree 1 and is extreme.

To complete the proof of our claim we show $\gamma_1 * \gamma_2 * \{x, y\}$ is the only extreme vertex in G. Since the number of extreme vertices in G is equal to the number of sub-equilibrium simplices lying in the boundary of $\mathcal{S}^*$ this is equivalent to showing the only sub-equilibrium face in the boundary of $\mathcal{S}^*$ is $\gamma_1 * \gamma_2 * \{x\}$.

Let Λ be a sub-equilibrium face on the boundary of $\mathcal{S}^*$. Then by Lemma 2.10 we can identify two simplices $C^{(1)}$ and $C^{(2)}$ such that $\Lambda = C^{(1)} * C^{(2)}$ where either $C^{(1)}$ belongs to $\mathcal{S}_{11}$ and $C^{(2)}$ belongs to $\mathcal{S}_{22}$ or vice-versa, so $C^{(1)}$ belongs to $\mathcal{S}_{12}$ and $C^{(2)}$ belongs to $\mathcal{S}_{21}$. Assume initially $C^{(1)}$ lies in $\mathcal{S}_{11}$ and $C^{(2)}$ lies in $\mathcal{S}_{22}$. Then by definition, and since Λ is sub-equilibrium, the vertex set of $C^{(1)}$ must satisfy the relation $\phi(v(C^{(1)})) \subset \{m_1+1, \dots, m-1\}$. Therefore the remaining vertices, $\{1, \dots, m_1\}$, must be contained in $C^{(2)}$ and equivalently $C^{(2)}$ must have Σ_{m_1} as a face. This contradicts the definition of $\mathcal{S}^*$.

As a result it must be that $C^{(1)}$ lies in $\mathcal{S}_{12}$ and $C^{(2)}$ lies in $\mathcal{S}_{21}$. Then $\phi(v(C^{(1)}))$ must be a *strict* subset of $\{1, \dots, m_1\}$ and the vertex labels $\{m_1 + 1, \dots, m - 1\}$ must belong to $\phi(v(C^{(2)}))$. Since $\dim \Lambda = \dim \mathcal{C}^{(1)} + \dim \mathcal{C}^{(2)} + 1 = (m - 2)$ we must have $\dim C^{(1)} = (m_1 - 2)$ and $\dim C^{(2)} = (m - m_1 - 1)$. We have identified $(m - m_1 - 1)$ of the vertices for $C^{(2)}$ and so there exists a unique vertex $w \in V \backslash V'$ such that $C^{(2)} * \{w\}$ lies in $\mathcal{S}_{21}$. Therefore $w \in W_1$. By our assumption Λ is a sub-equilibrium so $\phi(v(C^{(1)} * \{w\})) = \{1, \dots, m_1\}$. In particular $C^{(1)} = \gamma_1$ and $C^{(2)} = \gamma_2 * \{x\}$ as required.

Therefore there is a unique sub-equilibrium face on the boundary of $\mathcal{S}^*$ and thus there is a unique extreme node in G. From Lemma 2.22 this tells us there are an odd number of normal nodes in G and consequently there must be an odd number of equilibrium simplices in $\mathcal{S}^*$. Finally since the simplicial game complex $\mathcal{S}^*$ contains a finite number of vertices there can only be a finite number of equilibrium simplices.

$\square$

5.3 Simplicial Complexes of Order 2 and Lemke-Howson

Now we have a new formulation for simplicial game complexes of the order 2 we continue our discussion from Chapter 3.6. Observe the simplicial complexes $\mathcal{S}_{21}$ and $\mathcal{S}_{22}$ have a direct correspondence to the graphs F_1 and F_2 respectively (minus the artificial equilibrium point). In particular construct the nerve of these simplicial complexes. Then if a face of $\mathcal{S}_{21}$ is the convex hull of vertices with label set equal to K, in the nerve the same face becomes a vertex labelled by K. It is clear, after the addition of the artificial equilibrium point 0 as discussed in Chapter 3.6, the resulting nerve will satisfy the same definitions as F_1 and F_2. Therefore the procedure described by Shapley can be identically repeated here to achieve the same result in the same way as Lemke and Howson. This should come of no surprise that this is the case.

Remark
Of course instead of adding the artificial equilibrium we can identify the only sub-equilibrium simplex belonging to the boundary of $\mathcal{S}^*$.

5.4 Nash's Theorem for Bimatrix Games

Let η be the nerve constructed from the 2-player non-cooperative non-degenerate (traditional) game Γ. Then in order to determine the relationship between η and the new simplicial game complex $\mathcal{S}^*$ it is clear we need to define η as the join of two smaller simplicial complexes. From Chapter 4 recall in an N-player game Γ each player's polylinear payoff function generates a covering of connected sets over the bounded space $P^{(i)}$. For all $i = \{1, \ldots, N\}$ denote the nerve of this coverings (with boundary) by η_i. Then in Γ and for $i = \{1, 2\}$, η_i is the nerve of the covering over the bounded simplex P_j, $j = \{1, 2\}, j \neq i$ formed with respect to payoff function G_i. In this 2-player case we claim $\eta = \eta_1 * \eta_2$ and its proof is a simple observation. By definition the boundary of $\eta = \eta_1 * \eta_2$ is $\partial P_1 * \partial P_2$ which is the nerve of P. Further if Δ is a maximal simplex in η then the vertices in sets

$W'_{\Delta,i}$ and $\Sigma'_{\Delta,i}$ for $i = \{1,2\}$ can be associated to a unique simplex in $\Delta_j \subset \eta_j$ for a unique $j \in \{1,2\}$. Consequently η is equivalent to taking the nerve of the two coverings formed with respect to G_1 and G_2 over P_2 and P_1 respectively. Therefore $\eta = \eta_1 * \eta_2$ as required.

As before we now redescribe the nerve η using the terminology and notation used to define $\mathcal{S}^*$ in Definition 5.2. In particular for $i = \{1,2\}$ each η_i will be described as the simplicial complex $\mathcal{S}_{2i}$ and as such the bimatrix game will correspond directly to the simplicial complex $\mathcal{S}^* = \mathcal{S}_{21} * \mathcal{S}_{22}$.

The labelling of each vertex in η_i is achieved in the manner described in Chapter 4.3. In particular for any maximal simplex Δ in η_i we have $v(\Delta) = W'_{\Delta,i} \cup W'_{\Delta,j} \cup \Sigma'_{\Delta,i} \cup \Sigma'_{\Delta,j}$ where $W'_{\Delta,j} = \Sigma'_{\Delta,i} = \emptyset$, $W_{\Delta,i} \neq \emptyset$ and $\Sigma'_{\Delta,j}$ maybe empty. The vertices in $W'_{\Delta,i}$ correspond to covering elements $B_i(s_i^k)$ and as such are labelled by the appropriate $s_i^k \in S_i$. Any vertex belonging to $\Sigma'_{\Delta,j}$ represents a boundary face of P_j. As before, across each boundary face there exists a unique $s_j^l \in S_j$ which does not belong to the support of any mixed strategy the face represents. Label the face by this pure strategy. Consequently the labelling procedure for η_i ensures each vertex is assigned a unique label corresponding to strategies from the set $\{S_1 \cup S_2\}$. If $|S_i| = m_i$ then the size of this labelling set is $m_1 + m_2 = m'$. For the set of natural numbers $\mathbf{m}' = \{1, \ldots, m'\}$ there is a natural bijection ϕ' which maps the set $\{S_1 \cup S_2\}$ into $\mathbf{m}'$. In particular we may assume without loss of generality, $\phi'(S_i) = \{m_{i-1}+1, \ldots, m_i\}$ where $m_0 = 0$ and $m_2 = m'$. Observe since each player must have at least two strategies it follows that $|\mathbf{m_i}| \geq 2$ for $i = \{1,2\}$. Assume without loss of generality $i = 1$, then by definition the boundary of η_1 is identical to the boundary of the mixed strategy set P_2 and so $\phi'(v(\partial\eta_1)) = \phi'(v(P_2)) = \{m_1+1, \ldots, m'\}$. In line with Definition 5.2 assign $P_i = \Sigma_{m_i}$ then

$$\partial\eta_1 \;=\; \partial P_2 \;=\; \partial\Sigma_{m_2} \;=\; \mathcal{S}_{11} \tag{5.2}$$

as required. Equivalently $\partial\eta_2 = \mathcal{S}_{12}$ and the boundary of each η_i is identical to the boundary of the simplicial complex $\mathcal{S}_{2i}$. This process has lead to the natural partition of $\mathbf{m}'$ into sets $\mathbf{m_1} = \{1, \ldots, m_1\}$ and $\mathbf{m_2} = \{m_1+1, \ldots, m'\}$.

The set of vertices in η_1 which are not described by ∂P_2 represent player 1's optimal strategies for a given situation from P_2. Denote this set of vertices by W'_1 (note $W'_{\Delta,1} \subset W'_1$ for every maximal simplex $\Delta \subset \eta_1$). Player 1 must have at least one optimal strategy and therefore $W'_1 \neq \emptyset$. Further the labels associated to these vertices are a subset of $\mathbf{m_1}$ and therefore W'_1 satisfy the same properties as the set W_1 from Definition 3.1. Similarly let W_2' denote the set of interior points of η_2. We are then able to deduce the set of vertices V' is equal to $\phi'(S_1 \cup S_2)$ with the set of all vertices contained in η_1 and η_2 being $V = V' \cup (W_1') \cup (W_2')$.

Thus for $i = \{1, 2\}$ the nerve η_i can be described as our simplicial complex $\mathcal{S}_{2i}$. It is left to show each η_i also satisfies the following two additional properties required of $\mathcal{S}_{2i}$.

1. $\dim \eta_i = m' - m_i + m_{i-1} - 1$
2. η_i satisfies the conditions of non-ramification

In Lemma 5.1 we observed the nerve η over P for a 2 player game Γ is a non-ramified complex of dimension $m'-1$. In particular each player's payoff function generates a covering where all covering elements are convex polyhedra such that no two covering elements share the same label. Consequently it must also be the case that the nerves η_i, $i = \{1, 2\}$ are both non-ramified simplicial complexes.

Let Δ represent a maximal simplex in η where sets $\Sigma'_{\Delta,1}$ and $\Sigma'_{\Delta,2}$ are maximal. Then this intersection occurs at a vertex of the boundary and we must have $\dim P_i + \dim P_j$ boundary faces intersecting and 2 covering elements. In η_i this is equivalent to $\dim P_j + 1 = (m_j - m_{j-1} - 1) + 1 = m_j - m_{j-1}$ vertices. Observe

$$m_j - m_{j-1} \;=\; m' - (m_i - m_{i-1}) \;=\; m' - m_i + m_{i-1} \tag{5.3}$$

Therefore there exists a simplex in η_i of dimension $(m' - m_i + m_{i-1} - 1)$ as required. Repeating the argument in the proof of Lemma 4.16 determines this is the maximum dimension we can expect when Γ is non-degenerate. Consequently $\eta = \eta_1 * \eta_2$ is equivalent to Definition 5.2 and by Theorem 5.3 η contains an odd number of equilibrium simplices. Repeating the argument from Lemma 4.15, regarding the relationship between equilibrium situations and simplices, we can deduce there are finite and odd number of equilibrium situations in the bimatrix game Γ.

Remark

In this section we have returned to using the traditional game Γ. We believe the result does hold for the generalised game but additional work is needed to determine an appropriate "regularisation" process which will ensure the nerves η_1 and η_2 are non-ramified simplicial complexes.

5.5 Comparison

5.5.1 Simplicial Complexes for Bimatrix Games

We now appear to have two simplicial game complex representations for bimatrix games Γ, namely $\mathcal{S}$ and $\mathcal{S}^*$. Therefore we use this section to show in this case the two definitions coincide and in particular that $\mathcal{S}^*$ satisfies Definition 3.1.

The first two properties of $\mathcal{S}$ as given in Definition 3.1 automatically hold for $\mathcal{S}^*$ by Definition 5.2. We also know each $\mathcal{S}_{2i}$ is a non-ramified simplicial complex and so therefore $\mathcal{S}^*$ must be too. This just leaves the boundary, the subcomplex and ensuring $\mathcal{S}^*$ is of the correct dimension. We begin by addressing the latter.

The dimension of $\mathcal{S}_{2i}$ as given in Definition 5.2 is $(m - m_i + m_{i-1} - 1)$. Then by (5.1) the dimension of $\mathcal{S}^*$ is equal to $m' - 1$.

Therefore the dimension of $\mathcal{S}^*$ is equal to the required dimension of $\mathcal{S}$. Now consider the boundary. We observe that since both $\mathcal{S}_{21}$ and $\mathcal{S}_{22}$ are bounded simplicial complexes then $\mathcal{S}^*$ must also be bounded. For contradiction we assume this boundary does not satisfy the definition given for $\partial\mathcal{S}$. Let Δ be a $(m-1)$-simplex in $\mathcal{S}^*$ with a face, Λ, on the boundary where $\phi(\Delta) = W_{\Delta,1} \cup W_{\Delta,2} \cup \Sigma_{\Delta,1} \cup \Sigma_{\Delta,2}$ such that neither $W_{\Delta,1}$ or $W_{\Delta,2}$ contains a unique element. Recall sets $\Sigma_{\Delta,1}$ and $\Sigma_{\Delta,2}$ contain the boundary vertices of the simplex. Therefore if a boundary face exists it must be constructed by removing a vertex from either of the sets $W_{\Delta,i}$. Assume without loss of generality Λ is a boundary face of Δ which is constructed by removing a vertex with label from $\mathbf{m_1}$ so belongs to the set $W_{\Delta,1}$. Let the vertices of Λ be divided into two sets $W_{\Lambda,1}, W_{\Lambda,2}, \Sigma_{\Lambda,1}$ and $\Sigma_{\Lambda,2}$ where as in Definition 3.1 $W_{\Lambda,i}$ contains vertices from Λ belonging to W_i and $\Sigma_{\Lambda,j}$ contains the vertices from Λ belonging to Σ_{m_j}. By definition $|W_{\Delta,1}| \neq 1$. Since each simplex must contain a vertex from W_1 we must have $|W_{\Delta,1}| > 1$, and therefore the set $W_{\Lambda,1}$ is not empty. Since the vertices in $W_{\Lambda,1}$ are a strict non-empty subset of W_1 and $W_{\Lambda,2}$ is also a non-empty subset of W_2 the face Λ is the join of a simplex from $\mathcal{S}_{21}$ and a simplex from $\mathcal{S}_{22}$. This contradicts the conditions given in Lemma 2.10 to be a boundary face. Therefore if Λ is a boundary face of $\mathcal{S}^*$ when at least one of the sets $W_{\Lambda,1}$ and $W_{\Lambda,2}$ is empty. This just leaves the question of the subcomplex. However Definition 3.6 and Lemma 3.8 can be applied here to ensure the result.

We have now successfully shown the simplicial complex $\mathcal{S}^*$ satisfies the definition of the simplicial game complex $\mathcal{S}$ of the order 2.

5.5.2 Simplicial Complexes of the Order n

The definition of $\mathcal{S}^*$ to be a simplicial game complex can be extended to represent games with more than 2 players as follows. The preliminary description of $\mathcal{S}^*$ will be indistinguishable from that given for $\mathcal{S}$ preceding Definition 3.1. Let $\mathcal{S}_{1i} := \partial\Sigma_{m_1} * \cdots * \partial\Sigma_{m_{i-1}} * \partial\Sigma_{m_{i+1}} * \cdots * \partial\Sigma_{m_n}$ then define $\mathcal{S}_{2i}$ to be an arbitrary simplicial complex with vertex set $\phi(v(\mathcal{S}_{1i}))$ and W_i such that

1. The restriction of $\mathcal{S}_{2i}$ on to V' is $\mathcal{S}_{1i}$.
2. After the removal of some (of possibly zero) simplices from $\mathcal{S}_{2i} \setminus \mathcal{S}_{1i}$, the complex $\mathcal{S}_{2i}$ becomes an $(m - m_i + m_{i-1} - n + 1)$ dimensional non-ramified, bounded simplicial complex. If $\mathcal{S}_{2i}$ is already in this form then it is *non-degenerate*.

Then $\mathcal{S}^* := \mathcal{S}_{21} * \cdots * \mathcal{S}_{2n}$ and is non-degenerate if, for all $i = \{1, \ldots, n\}$, $\mathcal{S}_{2i}$ is non-degenerate.

Unfortunately, using the techniques from Chapter 3, the resulting simplicial game complex can **not** be shown to contain a finite and odd number of equilibrium simplices. In particular the resulting graph G fails to satisfy Lemma 2.22. Since this Lemma is the key concept of Nash's Theorem for simplicial game complexes this suggests the result is unlikely (but not proven impossible) to be achieved in this setting. To understand why we first need to consider the dimension of $\mathcal{S}^*$.

By definition:

$$\begin{aligned}
& \mathcal{S}^* &=& \quad \mathcal{S}_{21} * \cdots * \mathcal{S}_{2n} \qquad \text{for } n > 2 \\
\therefore \quad & \dim \mathcal{S}^* &=& \quad \dim \mathcal{S}_{21} + \cdots + \dim \mathcal{S}_{2n} + (n-1) \\
& &=& \quad (m - m_1 + m_0 - n + 1) + \cdots + (m - m_n + m_{n-1} - n + 1) + (n-1) \\
& &\gg& \quad m - 1 \qquad \text{for } n > 2 \qquad (5.4)
\end{aligned}$$

Our requirement that $\mathcal{S}^*$ satisfies the condition of non-ramification only provides information about those simplices in $\mathcal{S}^*$ which are of maximal dimension. In particular the property of non-ramification ensures each simplex of maximal dimensions shares a face with at most one other maximal simplex; equivalently each face can belong to at most two simplices. This property does not hold for simplices in $\mathcal{S}^*$ which are of smaller dimension. Figure 5.1 illustrates this by showing a simplicial complex of dimension 2.

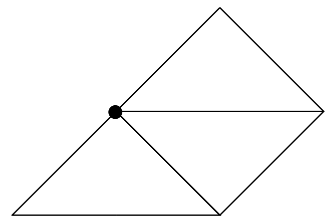


Figure 5.1: 2-dimensional non-ramified simplicial complex. Note that more than 2 edges intersect at a point as non-ramification does not affect faces of this dimension.

In Figure 5.1 each 1-dimensional face within the complex belongs to at most 2 simplices and so the conditions of non-ramification are satisfied, yet the highlighted vertex or 0-dimensional simplex belongs to four 1-dimensional simplices and three 2-dimensional simplices. This can be extended to any finite dimension.

When $n = 2$ the dimension of $\mathcal{S}^*$ is $(m-1)$. Therefore the equilibrium and sub-equilibrium simplices under consideration are maximal simplices in $\mathcal{S}^*$ and so their intersections within the complex are restricted by the property of non-ramification. As such the vertices in the graph G, as constructed in the proof of Theorem 3.11, will be of degree less than or equal to two. When $n > 2$ equation (5.4) tells us the dimension of $\mathcal{S}^*$ is now in excess of $(m-1)$. Therefore the equilibrium and sub-equilibrium simplices of $\mathcal{S}^*$ are no longer maximal and the corresponding graph G will contain vertices of degree strictly greater than two. Consequently we are unable to use Lemma 2.22 and the result is undetermined.

We return briefly to the nerves η_i constructed from the coverings over the bounded sets $P^{(i)}$ in the N-player generalised games. Observe in this case $\eta_1 * \cdots * \eta_N$ is a simplicial complex larger than η as defined in Chapter 4. This causes additional complications as the simplicial complex $\eta_1 * \cdots * \eta_N$ will contain simplices representing situations which are not realisable in the game (i.e., situations which do not represent a situation defined by the N coverings or contained within P). Similarly within the simplicial game complexes when $n > 2$ we observe $\mathcal{S} \subset \mathcal{S}^*$ which coincides with the properties of the nerves.

Chapter 6

Degenerate Games

Our work so far has been restricted to examining the behaviour of the generalised games and simplicial complexes which satisfy the condition to be non-degenerate. However we recall Nash's Theorem, as given by John Nash in 1950 [Nash, 1950b], is concerned with proving the existence of an equilibrium situation in any non-degenerate **and** degenerate game. As such, while our work has extended Nash's Theorem in the non-degenerate case, the degenerate case still needs to be answered. Therefore to ensure completeness we use this Chapter to discuss degeneracy firstly in terms of the simplicial game complex model and then generalised games Γ^*.

6.1 Simplicial Complexes

By definition, the simplicial complex $\mathcal{S}$ is non-degenerate if it is of dimension $(m-1)$ and if it satisfies the properties of non-ramification. Consequently if $\mathcal{S}$ is degenerate then condition 4 from Definition 3.1 is not met. In such cases we must be able to remove a finite number of simplices to generate a subcomplex of $\mathcal{S}$ which satisfies all of Definition 3.1 and in particular is non-degenerate. Example 6.1 demonstrates this process.

Example 6.1
Let $\mathcal{S}$ be a simplicial complex of order 1, as defined in Definition 3.1, except with regards to the property of non-ramification, constructed with respect to set $\mathbf{m} = \{1, 2, 3\}$. Label the vertices of $\mathcal{S}$ by $\{a, \ldots, g\}$ where each label corresponds to a unique value from $\mathbf{m}$ such that Definition 3.1 is satisfied. Then, by definition, if $\mathcal{S}$ is non-degenerate then it is

of dimension 2 and is a non-ramified simplicial complex. Figure 6.1 shows a segment of $\mathcal{S}$ transformed from being degenerate to non-degenerate.

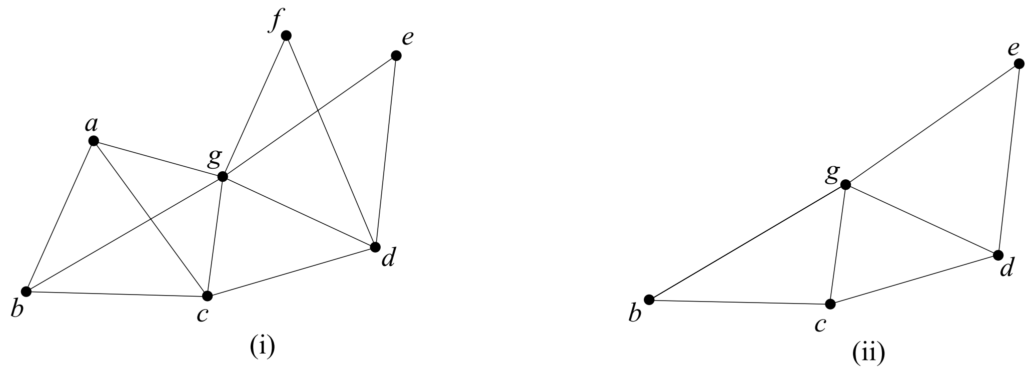


Figure 6.1: Transforming a simplicial complex $\mathcal{S}$ from degenerate to non-degenerate.

Observe simplicial complex (i) in Figure 6.1 is degenerate on two accounts; firstly simplex $\{a, b, c, g\}$ is of dimension 3 and not 2 and the face $\{g, d\}$ belongs to 3 simplices and thus fails the condition of non-ramification. Complex (ii) shows a non-degenerate simplex which has been attained by removing vertices $\{a\}$ and $\{f\}$ from complex (i), this is equivalent to removing simplices

$$\begin{array}{ll} \{a, b\} & \{f, d\} \\ \{a, c\} & \{f, g\} \\ \{a, g\} & \end{array}$$

However these are not the only vertices we could remove to achieve an appropriate complex, removing vertex $\{b\}$ instead of $\{a\}$ and/or $\{e\}$ instead of $\{f\}$ would also be sufficient.

Remark

Removing vertices $\{a\}$ and $\{f\}$ from complex (i) are the minimum which need to be removed to ensure $\mathcal{S}$ is non-degenerate. However, if (i) shows the entire simplicial complex $\mathcal{S}$, removing $\{b\}$ and/or $\{e\}$ in addition to these vertices will also result in the complex being non-degenerate. Observe removing any vertex from $\{c, d, g\}$ will have the reverse affect.

(End Example)

Recall a degenerate simplicial game complex $\mathcal{S}$ can be identified by simplices containing too many vertices or by faces belonging to too many simplices. Therefore we must always be able to successfully identify a non-degenerate sub-complex by removing a finite number of simplices.

By Lemma 3.11 we know any non-degenerate complex $\mathcal{S}$ contains a finite and odd number of equilibrium simplices, in particular it must contain at least one. Then since in every degenerate complex $\mathcal{S}$ a non-degenerate subcomplex $\bar{\mathcal{S}}$ can be identified, $\mathcal{S}$ must contain at least as many equilibrium simplices as $\bar{\mathcal{S}}$. Therefore there is at least one equilibrium simplex in $\mathcal{S}$.

This leads us to the full version of Nash's Theorem for simplicial game complexes.

Theorem 6.2 (Nash's Theorem for Simplicial Game Complexes)
The simplicial game complex $\mathcal{S}$ contains at least one equilibrium simplex. When $\mathcal{S}$ is non-degenerate the number of equilibrium simplices is finite and odd.

Remark
The conditions for the simplicial complex $\mathcal{S}_{2i}$ from Definition 5.2 to be degenerate are identical to those for $\mathcal{S}$. Therefore the above result and reasoning also applies to these simplicial complexes.

6.2 Non-Cooperative Generalised Games

Focus now returns to game theory and we work to extend Theorem 4.24, Generalised Nash's Theorem for non-degenerate games Γ^*, to include the degenerate case. Previous discussions within this Thesis allows us to be assured all traditional degenerate games Γ are contained in the set of all generalised degenerate games Γ^*.

By Definition 4.17 a nerve η of a degenerate game will fail at least one of the following conditions.

1. $\dim \eta = m' - 1$

2. Locally each intersection in the covering corresponds to a simplex satisfying the conditions of non-ramification.

3. The process of regularisation of the entire simplicial complex nerve η results in a non-ramified simplicial complex.

The following examples demonstrate the first two of these conditions failing.

Example 6.3
Consider a traditional degenerate bimatrix dyadic game and assume the game is degenerate with respect to player 1. Figure 6.2 shows the best response function for player 1.

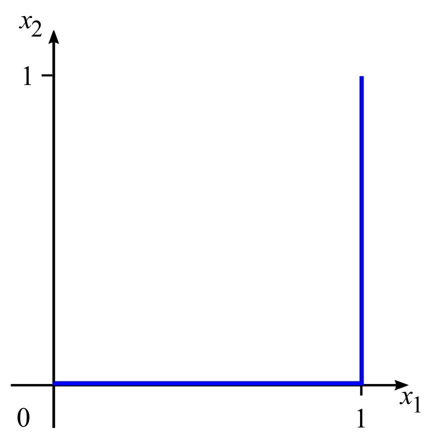


Figure 6.2: Degenerate best response function

Figure 6.3 shows the corresponding nerve η_1.

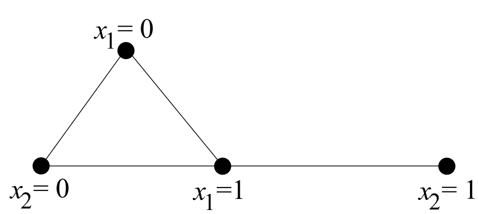


Figure 6.3: Nerve η_1 of a degenerate game

Recall from Definition 5.2 in a non-degenerate game we expect the dimension of the nerve η_1 to be

$$(m' - m_i + m_{i-1} - 1) \quad = \quad (4 - 2 + 0 - 1) \quad = \quad 1 \tag{6.1}$$

Then the nerve η_1 in Figure 6.3 is not of the correct dimension. However a slight perturbation of the original covering elements can result in the best response function shown in Figure 1.6 with resulting nerve in Figure 6.4

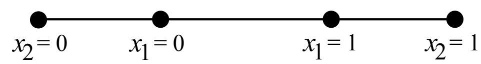


Figure 6.4: Nerve of a degenerate game after pertubation

(End Example)

Example 6.4
The intersection of covering elements seen in Figure 6.5 corresponds to an intersection in a 3-player degenerate dyadic game.

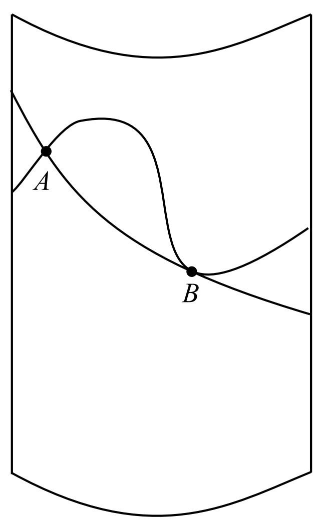


Figure 6.5: Degenerate intersection of covering elements

The intersection marked A is transverse where the intersection marked B is not. Let Δ represent the intersections A and B in the nerve. Observe the simplex Δ satisfies the conditions of non-ramification therefore, it appears η is non-degenerate and nothing need to be done. However in this instance Δ represents two situations and the total number of equilibrium simplices in η is neither equal to or of the same parity as the number of equilibrium situations in the game. This is a direct result of the covering failing property

2 at B.

As in Example 6.3 a slight perturbation of the covering elements removes the degenerate intersection, such that either intersection B is removed from the covering completely or 1 additional intersection is introduced. This corresponds to transforming the degenerate game into a non-degenerate one with one less or one additional equilibrium situation.

(End Example)

The intersection of covering elements in a degenerate game can always be slightly perturbed to ensure all intersections are transverse and as such every degenerate game can always be transformed into at least one non-degenerate game.

Suppose perturbing the covering elements in a degenerate game results in a non-degenerate game with fewer intersections corresponding to equilibrium situations. Consequently the nerve of the non-degenerate transformation will be a subcomplex of the nerve of the original degenerate game and, in particular, the degenerate game will contain at least as many equilibrium situations as the non-degenerate game it has been transformed into.

Alternatively assume the number of equilibrium intersections in the non-degenerate game is greater than or equal to the number in the original degenerate game. Then the non-degenerate nerve will need to be regularised (of course we cannot discount this from being a possibility in the first case either) which will have the affect of reducing the number of intersections in the non-degenerate covering. Consequently, once again the regularised non-degenerate nerve will be a subcomplex of the degenerate one.

Remark
If the translation from degenerate into non-degenerate introduces new intersections within the covering then in the nerve this has the affect of increasing the number of simplices shared by a face and hence affects the conditions of non-ramification. Regularisation is therefore required and each application of this process will reduce the number of intersections occurring at the ramified face by 2. This can be repeated for intersections occurring in any finite number of connected components and as such regularisation will always reduce the number of intersections to a number smaller than found in the original degenerate game.

Finally, we arrive at

Theorem 6.5 (Generalised Nash's Theorem)
All generalised games Γ^* *have at least one equilibrium situation. When* Γ^* *is non-degenerate the number of equilibrium situations is finite and odd.*

Remark
Compare this to Lemke and Howson's original constructive procedure [Lemke and Howson Jr, 1964] (Shapley does not discuss degenerate games). Then a small perturbation to the convex polyhedra, defined by Lemke and Howson, describing a degenerate game will result in a large enough distortion to identify a non-degenerate 'sub-game'. That is the resulting convex polyhedra represents a non-degenerate game where the total number of equilibrium situations is at most the number contained in the original degenerate game. In our representations, this is equivalent to a small shift in covering elements to ensure all intersections are transverse.

Chapter 7

Consequences and Examples of Generalised Games

The objectives for this Thesis as set out in Chapter 3 have now been fulfilled. We use this Chapter to discuss the consequences of our results and provide some general examples of its application.

7.1 The Consequences of a Generalised Result

The key to our result lies in the representation of each player's optimal strategies, firstly as a covering and then as a simplicial complex. Using this approach enabled us to gain an insight to a generalised model which contained a suitable formalisation of Nash's Theorem. This afforded us the ability to prove the original formulation of Nash's Theorem combinatorially. In particular our work justifies Nash's Theorem is not a special consequence of polylinear payoff functions, indeed we have gone further and shown that Nash's Theorem is not a result dependent on the properties exhibited by a payoff function, but rather is a specific example of a more general model.

The proof of Theorem 3.11 demonstrates the existence of a general form of Nash's Theorem in the simplicial game complex $\mathcal{S}$. This combined with our result that those simplicial complexes arising from game theory are examples of $\mathcal{S}$ allow us to draw the following conclusion.

> We have successfully provided a proof of Nash's Theorem without the need for any geometric considerations. In particular we have shown Nash's Theorem can be proved using an abstract version of Sperner's Lemma. Additionally we have defined a simplicial complex model for which an analogy of Nash's Theorem exists. This has allowed Nash's Theorem to be proved for a class of abstract games where the payoff function is no longer required to be an expectation.

The consequences of Theorem 6.5 for game theory lies in the ability to define and construct more complex and abstract games for which Nash's Theorem will be applicable. As such we have demonstrated a larger category of games will be guaranteed to contain at least one equilibrium situation.

Throughout this Thesis we have also compared our result to the work by Lemke and Howson [Lemke and Howson Jr, 1964]. In particular the method of identifying an equilibrium situation arising from our combinatoric proof not only replicates the description of the constructive procedure provided by Shapley [Shapley, 1974] but also generalises it. In particular we have shown the paths described by Shapley appear in our generalised games and simplicial game complexes.

An important point to observe is the ability to provide a refinement to the definition of the simplicial complex of order n for the case $n = 2$. This mimics the behaviour of game theory where bimatrix games afford simplifications over the general N-player (and even 3-player) case. This suggests the added complications introduced when $N > 2$ is also a result of the underlying mathematical model and not a direct consequence of the game theory formulation.

7.2 Examples

We provide some examples of generalised non-cooperative games which satisfy Nash's Theorem. We will include games with non-polylinear payoff functions and games with different input strategies.

Remark
The term *game* now extends to our generalised games Γ^*, Definition 4.4, and therefore is a broader definition than given previously. As such the games given as examples in this Section will be of an abstract nature.

7.2.1 Payoff Function

In this Thesis we have proven the polylinear payoff function G_i can be replaced by a total order $\succ_i$. However in Chapter 3 we observed certain games would still require a payoff value and in such circumstances the use of a total order would be too restrictive. Therefore in this section we provide examples of games defined by payoff functions and total orders. Naturally all examples in this section will satisfy Theorem 6.5, Nash's Theorem for Generalised Games.

To ensure simplicity in representation, all examples in this section will be for bimatrix games. For $i = \{1, 2\}$, if $|S_i| = l_i$ then let $S_i = \{s_i^1, \ldots, s_i^{l_i}\}$ represent the set of pure strategies and $p_i = (x_i^1, \ldots, 1 - (x_i^1 + \ldots + x_i^{(l_i-1)})) \in P_i$ be a mixed strategy situation for player i.

Example 7.1

To enable easy comparison, this first example is the traditional bimatrix dyadic game where the payoff functions are polylinear. In this case the payoff function assigned to player 1 can be written generally as:

$$G_1 : P_1 \times P_2 \quad \mapsto \quad x_1^1(a_1 x_2^1 + a_2(1 - x_2^1)) \;\; + \;\; (1 - x_1^1)(a_3 x_2^1 + a_4(1 - x_2^1)) \tag{7.1}$$

Without loss of generality assume $a_3 < a_2 < a_1 < a_4$. In order to construct the coverings discussed in Chapter 4, for all situations $\mathbf{p} \in P$, we consider the restricted payoff functions of G_1. This results in 2 payoff functions, each defining straight lines. These functions are given by:

$$G_1^1 \; = \; G_1(\mathbf{p}, s_1^1) : \{s_1^1\} \times P_2 \; \mapsto \; a_1 x_2^1 + a_2(1 - x_2^1) \tag{7.2}$$

$$G_1^2 \; = \; G_1(\mathbf{p}, s_1^2) : \{s_1^1\} \times P_2 \; \mapsto \; a_3 x_2^1 + a_4(1 - x_2^1) \tag{7.3}$$

Graph (i) in Figure 7.1 represents equation (7.2) and graph (ii) represents equation (7.3).

Combining the graphs from Figure 7.1 onto the same axis, as seen in Figure 7.2, allows easy identification of player 1's optimal strategies for all situations in P_2.

We can now translate this information to generate a covering of the simplex P_2. Since we are considering a bimatrix dyadic game, P_2 is a 1-dimensional simplex. Figure 7.3 shows

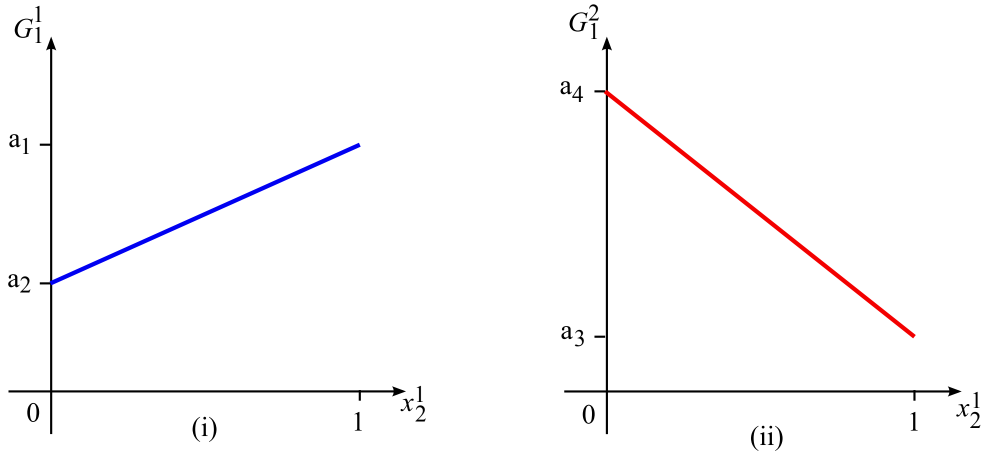


Figure 7.1: Player 1's restricted payoff functions as given in (7.2) and (7.3)

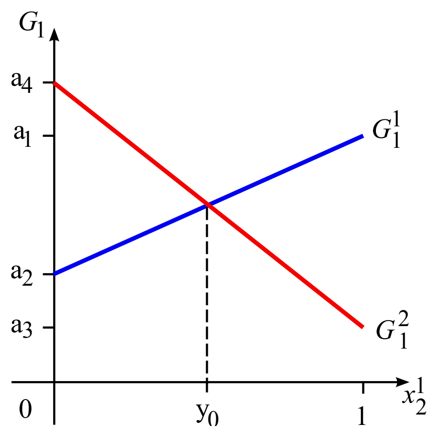


Figure 7.2: Determining optimal strategies by comparing the two restricted payoff functions

the covering of P_2 formed with respect to the information contained within Figure 7.2. For the set of mixed strategies $\{(x_2^1, 1-x_2^1) \,|\, 0 \leq x_2^1 < y_0\}$ player 1's optimal strategy is $x_1^1 = 0$ i.e., s_1^2 is optimal. When player 2 uses any mixed strategy from the set $\{(x_2^1, 1-x_2^1) \,|\, y_0 < x_2^1 \leq 1\}$ player 1's optimal strategy is s_1^1. Finally when $x_2^1 = y_0$ player 1's optimal strategy is a totally mixed strategy situation. Each segment in Figure 7.3 is labelled by player 1's optimal strategy.

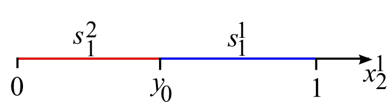


Figure 7.3: The resulting covering over P_2

Repeating this process for player 2 produces a covering similar to that seen in Figure 7.3, where by similar we mean consists of just two distinct covering elements. Then the two coverings over $P = P_1 \times P_2$ is shown in Figure 7.4. Each segment in Figure 7.4 is labelled by the optimal strategies for player 1 and 2.

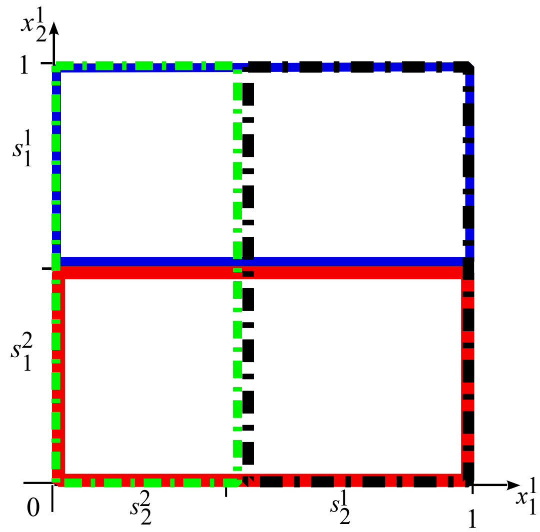


Figure 7.4: The resulting covering over P

(End Example)

The simple division of the simplex into two segments (as seen in Fig 7.3) is the most that can be achieved using polylinear functions (in the bimatrix dyadic game). Equivalently this is the most complex a game could be while ensuring Nash's Theorem holds. We now give some examples which demonstrate the power of Theorem 4.24 in extending Nash's Theorem to much more advanced games.

Example 7.2

Let Γ^* be a generalised bimatrix dyadic game where for player 1

$$G_1 : P_1 \times P_2 \quad \mapsto \quad x_1^1(\cos(4\pi \times x_2^1 + \pi) + 1) \; + \; (1 - x_1^1)(\cos(4\pi \times (1 - x_2^1)) + 1) \quad (7.4)$$

First observe this function satisfies our requirements of a total order, Definition 4.3. In particular, for any value x_2^1 the function G_1 becomes linear in x_1^1 and therefore a maximum must occur at at least one of the pure strategies. If both pure strategies are maximum then the function has 0 gradient and therefore all mixed strategies are best responses.

The two restricted payoff functions are given in (7.5) and (7.6) and are shown in Figure 7.5

$$G_1^1 \;=\; G_1(\mathbf{p}, s_1^1) : \{s_1^1\} \times P_2 \mapsto \cos(4\pi \times x_2^1 + \pi) + 1 \tag{7.5}$$

$$G_1^2 \;=\; G_1(\mathbf{p}, s_1^2) : \{s_1^2\} \times P_2 \mapsto \cos(4\pi \times (1 - x_2^1)) + 1 \tag{7.6}$$

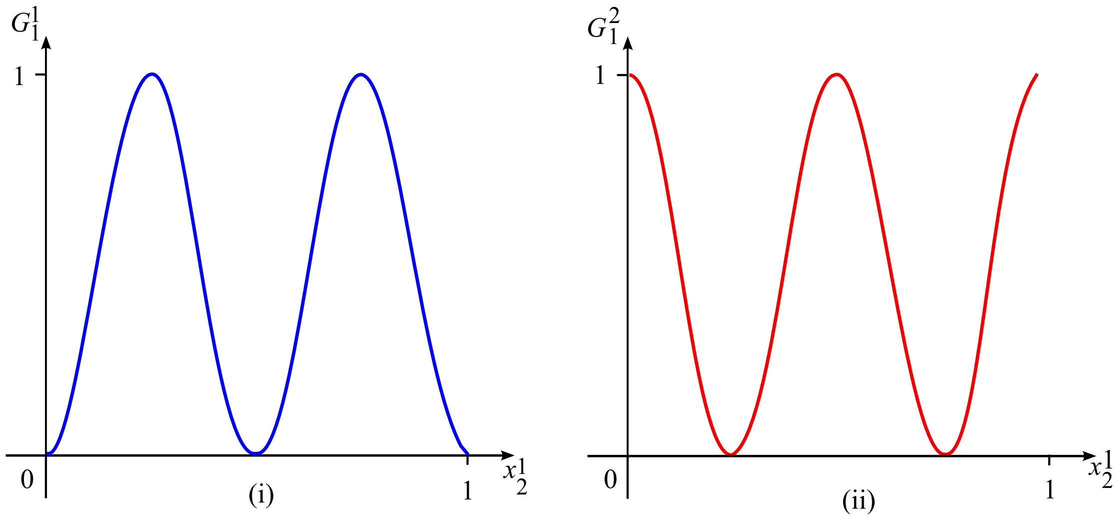


Figure 7.5: Player 1's restricted payoff functions as given in (7.5) and (7.6)

We now compare the graphs of the restricted payoff functions on the same set of axis. This allows for easy comparison and subsequent construction of the required covering. This is shown in Figure 7.6

For all $p_1 \in P_2$, Figure 7.6 identifies player 1's optimal pure strategies which in turn produces the covering given in Figure 7.7. Each element of the covering is labelled by player 1's optimal strategy.

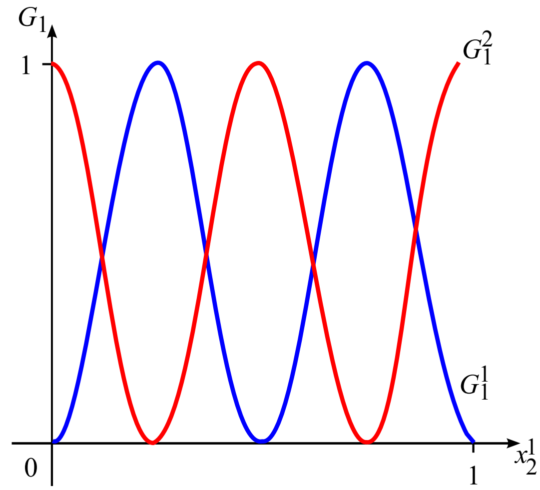


Figure 7.6: Determining optimal strategies by comparing the two restricted payoff functions

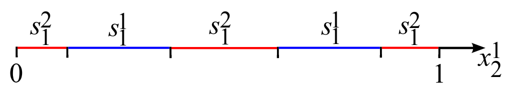


Figure 7.7: The resulting covering over P_2

Where the nerve of the covering given in Figure 7.7 is as seen in Figure **??**.

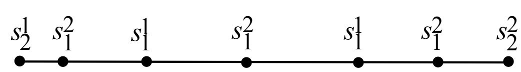


Figure 7.8: The nerve for the covering given in Figure 7.7

Define a payoff function for player 2 which produces the same results, then Figure 7.9 shows the two coverings for the two players over P. In Figure 7.9 those strategies in boxes denote the labels of the boundary faces. We know equilibrium situations occur when there is complete labelling, or in this case all 'colours' are present at the intersection. In this game there are 13 equilibrium situations, all are marked in Figure 7.9.

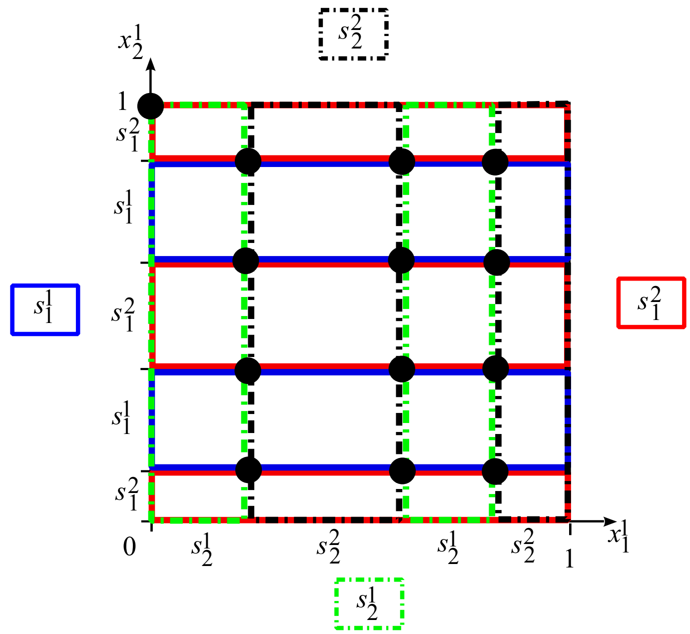


Figure 7.9: The resulting covering over P and the equilibrium situations of the game

We return to the constructive procedure contained in Lemke-Howson as described by Shapley [Shapley, 1974]. As discussed in Chapter 4.4 and Chapter 5.3, the coverings over P_1 and P_2 (with the additional vertex (0,0)) are equivalent to the graphs F_1 and F_2 defined by Shapley. These graphs are shown in Figure 7.10 and show a path starting from the artificial equilibrium point (0,0) and terminating with an equilibrium situation. From $(0,0)$ pure strategy s_2^2 is chosen as the label to be dropped. In the order in which they are visited, the path consists of the following nodes $(0,0), (A,0), (A,b), (C,b), (C,d)$.

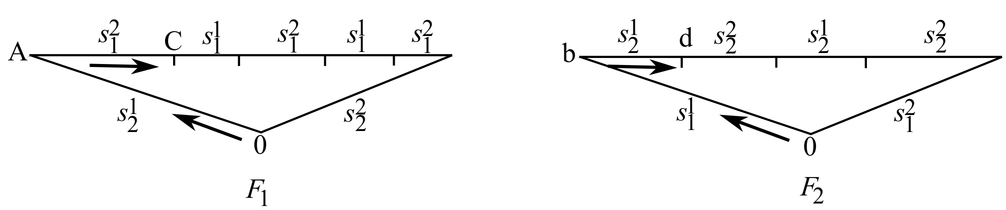


Figure 7.10: Graphs F_1 and F_2

This path only be represented on Figure 7.9 once it has moved completely away from the point $(0,0)$, as this is not represented in this Figure. Therefore the path starting at (A,b) and progressing through to (C,b) and finally terminating with the equilibrium point represented by (C,d) is illustrated in Figure 7.11.

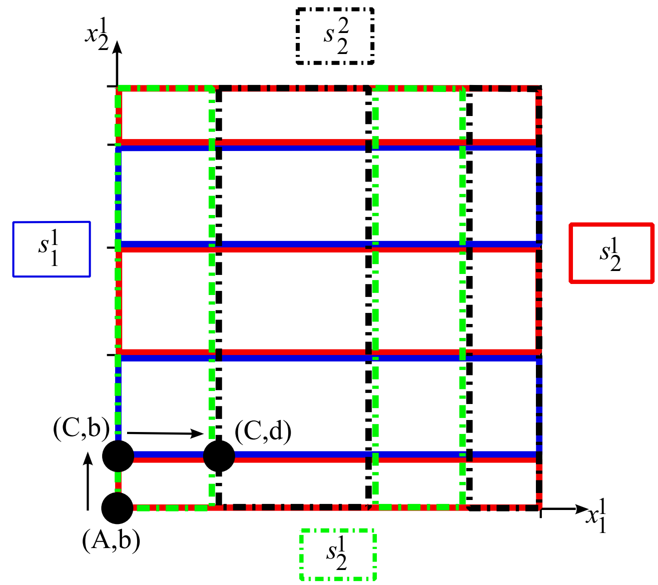


Figure 7.11: Example Lemke-Howson path

(End Example)

The advantage in being able to use more generalised functions is clear to see when comparing the covering of the original payoff function in Example 7.1 to the one generated in this Example 7.2 (Figures 7.4 and 7.9). Our generalisations allow games to use payoff functions which allow a greater variation in optimal strategies, especially with regards to the number of connected covering elements a given strategy can be optimal across, while still guaranteeing the existence of an equilibrium point. Previously to ensure the existence of an equilibrium situation the payoff functions of the game had to be polylinear. This is no longer the case. We continue with some more examples of where Nash's Theorem was previously unable to be used.

Example 7.3
Once again let Γ^* be a generalised bimatrix dyadic game. However this time we assume a total order (defined by any means) defines the covering elements over P as

$$B_1(s_1^1) = \left[0, \frac{2}{3}\right] \times [0,1] \tag{7.7}$$

$$B_1(s_1^2) = \left[\frac{1}{3}, 1\right] \times [0,1] \tag{7.8}$$

$$B_2(s_2^1) = [0,1] \times \left[0, \frac{2}{3}\right] \tag{7.9}$$

$$B_2(s_2^2) = [0,1] \times \left[\frac{1}{3}, 1\right] \tag{7.10}$$

Then this covering takes the form seen in Figure 7.12.

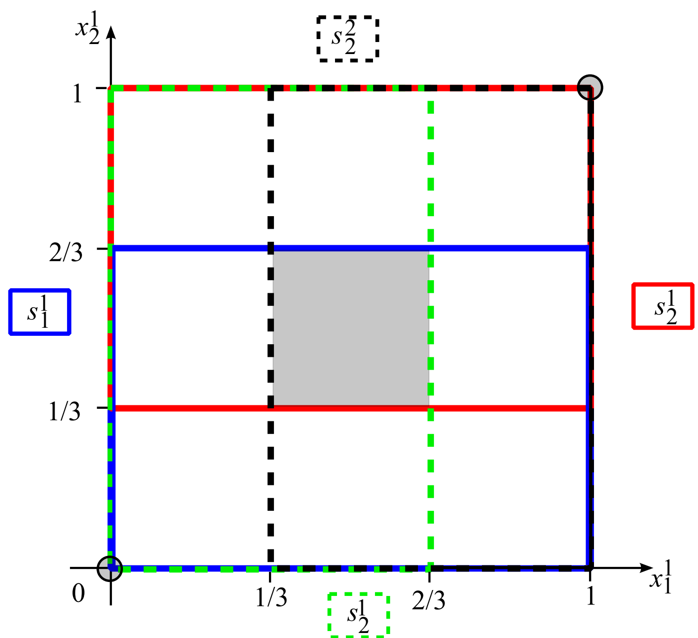


Figure 7.12: Coverings over P

Those labels occurring away from the covering indicate the labels of the corresponding boundary faces, while the shaded areas mark the equilibria of this game. The set of equilibrium points consists of two isolated points $(0,0)$ and $(1,1)$ and the square $\left[\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right] \subset P$.

Observe in games of this nature constructing the Lemke-Howson path, as in Example 7.2 is not applicable in its current form. However it is easy to check the nerve of the covering (with boundary) satisfies Definition 3.1. Then within the simplicial complex setting a path of sub-equilibrium and equilibrium simplices can be constructed inline with the Lemke-Howson algorithm.

(End Example)

Example 7.4
Let Γ and Γ^* be a bimatrix games where $|S_1| = 2$ and $|S_3| = 3$ then a generic best response correspondence with respect to payer 2 in the traditional case can be seen in (a) of Figure 7.13 and the best response correspondence in (b) is an example of a generalised game.

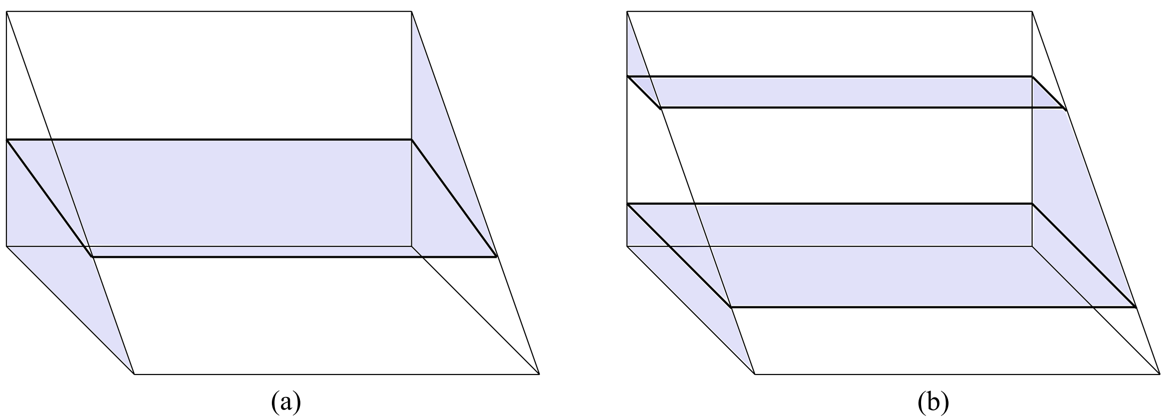


Figure 7.13: best response correspondence over P for Γ and Γ^* with respect to player 2

(End Example)

Example 7.5
All coverings seen in the previous example are those arising from generalised bimatrix games. However our proofs also apply to coverings which do not arise naturally from games. Consider the coverings over $[0, 1] \times [0, 1]$ given in Figure 7.14

As before, those labels which appear away from the covering refer to the labels of the boundary faces but note since this covering does not reflect a game we are not using the notation seen previously. The covering shown in Figure 7.14 consists of two coverings over the space $[0, 1] \times [0, 1]$. The first divides the space into two sets 'horizontally' along the edge

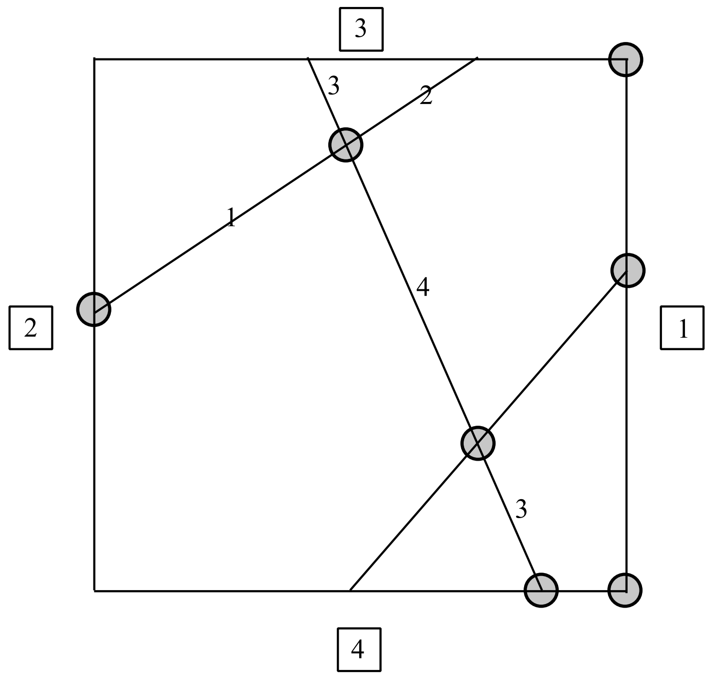


Figure 7.14: Coverings over $[0,1] \times [0,1]$ formed without relation to games

going from boundary face 3 to 4 and are labelled by 1 and 2. The second covering divides $[0,1] \times [0,1]$ into three segments, labelled by 3, 4 and 3 via the two remaining edges. Now observe the nerve can be constructed as before and will also satisfy definition 3.1. Then the shaded circles highlight the 7 equilibrium points of the covering.

Remark
While this example was constructed as an arbitrary covering, similar coverings can be seen in [von Stengel et al., 2002] as a representation of a extensive form game.

(End Example)

7.2.2 Input Strategies

We now turn our attention to a different consequence of our result. Not only does our work allow for the generalisation of the payoff functions but also for a less restrictive approach to the strategy sets. The following example is intended as an insight to the opportunities and benefits our Theorem can provide. For a bimatrix dyadic game we denote the mixed strategy set for player 1 by the 1-simplex P_1. However we assume the mixed strategy set for player 2 is the union of two 1-dimensional simplices or the set of real numbers $[0, z_1] \cup [z_2, 1]$

for $z_1, z_2 \in (0,1)$. Observe if $z_1 = z_2$ then the mixed strategy set for player 2 is the usual 1-dimensional simplex. To denote this distinction we will call this mixed strategy set $\hat{P}_2$. The total order $\succ_1$ for player 1 then identifies player 1's preferred strategy for all strategies from $\hat{P}_2$.

Remark

We still require player 2's mixed strategy situation to satisfy the conditions of a probability distribution

Figure 7.15 shows an arbitrary covering over sets P_1 and $\hat{P}_2$ where each segment is labelled by player 1 and player 2's optimal pure strategy for the corresponding strategies from $\hat{P}_2$ and P_1 respectively.

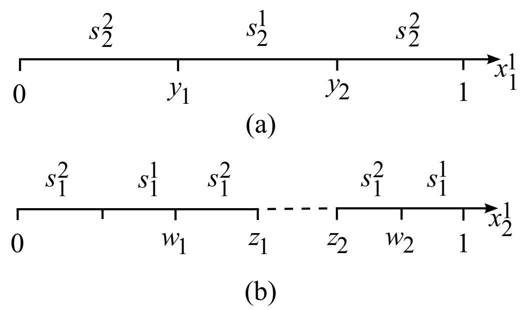


Figure 7.15: Coverings across (a) P_1 and (b) $\hat{P}_2$

Let the 1-dimensional segment $[0, z_1]$ of $\hat{P}_2$ be denoted by L_1 and segment $[z_2, 1]$ by L_2 with corresponding nerves η_{2_1} and η_{2_2}. Assume the optimal pure strategy for player 1 when player 2 selects strategy z_1 is the same when player 2 selects strategy z_2. Then we claim the simplicial complex defined by $(\eta_1 * \eta_{2_1}) \cup (\eta_1 * \eta_{2_2})$ represents a simplicial complex which satisfies the definition of $\mathcal{S}$ where the construction of the union is to be described.

First consider the boundary of the simplicial complexes. Previously the boundary faces of the nerve η represent the situations from P which are not totally mixed; in this case this is equivalent to one player selecting a pure strategy. Observe the coverings over L_1 and L_2 are not bounded in this sense. In particular since the strategy for player 2 at z_1 (respectively z_2) is totally mixed the boundary point represents a strategy for player 1 and not player 2. Figure 7.16 describes all simplices from $(\eta_1 * \eta_{2_1})$ and $(\eta_1 * \eta_{2_2})$ which are affected by this. The column labelled P_1 list the vertices from η_1 and for $i = \{1, 2\}$ the columns labelled L_i lists the vertices from η_{2_i}.

Simplex Number	P_1	L_1	L_2
1	$[0, y_1]$	$[w_1, z_1]$	
2	$[0, y_1]$		$[z_2, w_1]$
3	$[y_1, y_2]$	$[w_1, z_1]$	
4	$[y_1, y_2]$		$[z_2, w_1]$
5	$[y_2, 1]$	$[w_1, z_1]$	
6	$[y_2, 1]$		$[z_2, w_1]$

Figure 7.16: Simplices on the 'boundary' of $(\eta_1 * \eta_{2_1})$ and $(\eta_1 * \eta_{2_2})$

By Definition 3.1, the face of simplex 1 and 2 which should belong to the boundary is the one formed by removing vertex y_1. However on the assumption z_1 and z_2 are to be interpreted as boundary faces, and not covering elements, a second face can be identified which also satisfies the boundary definition. This face is the result of removing either z_1 or z_2 and leaves the face $\{0, y_1, x_1\}$ for both simplices. This can be repeated for the 2 other pairs $\{3, 4\}$ and $\{5, 6\}$. Therefore the simplicial complex $\eta = (\eta_1 * \eta_{2_1}) \cup (\eta_1 * \eta_{2_2})$ is formed by connecting $(\eta_1 * \eta_{2_1})$ to $(\eta_1 * \eta_{2_2})$ via the shared 'mis-defined' boundary faces. This result in a simplicial complex with a boundary, which by repeating Lemma 4.21, satisfies Definition 3.1. Additionally, for this example, the nerve η already satisfies the conditions of non-ramification.

We finally observe the vertices in η_1, η_{2_1} and η_{2_2} can all be labelled following the rules of Section 4.3 with vertices z_1 and z_2 taking on the roles of boundary faces. Since z_1 and z_2 are real numbers, this introduces labels which are not contained within $\{S_1 \cup S_2\}$, but once again in the construction of η no vertex is left without a label. In the construction of η we observe all 'mis-defined' boundary faces are connected by removing the vertices z_1 and z_2 and since these vertices only appear at these points then η does not contain any vertex labelled by z_1 or z_2. As before those simplices which are completely labelled in η must correspond to equilibrium situations from the game, and all equilibrium situations must be represented in this way. Therefore $(\eta_1 * \eta_{2_1}) \cup (\eta_1 * \eta_{2_2})$ is a particular example of $\mathcal{S}$ as defined in Definition 3.1 and consequently Theorem 4.24 assures there are a finite and odd number of equilibrium simplices and thus equilibrium situations in the game.

Chapter 8

Comparing New to Old

We compare our approach to prove Nash's Theorem with those discussed in Section 1.7. This will provide a detailed analyse of the differences and similarities our work has to previous papers. The similarities provided us with a path our proof may, and indeed would, follow, while the differences highlight our intuition in developing the result further.

The work published by Nash in 1950 and 1951, [Nash, 1950b] and [Nash, 1951] respectively, focuses only on proving the existence of equilibrium situations. Nash's use of a fixed point theorem was more than sufficient to allow him to produce two short proofs for this result. However fixed point theorems will not tell us anything about the cardinality of the 'fixed point set' as shown in the later proofs of Nash's result. Nash made use of Brouwer's fixed point theorem which can be proved by using Sperner's Lemma and a convergent sequence. We believe the appearance of Sperner's Lemma here as well as in our work is unrelated. Sperner's Lemma in relation to Nash's proof is used as a tool to help identify an approximate equilibrium situation (albeit a very good approximation). In our work Sperner's Lemma is used in its abstract form and it's its underlying combinatoric properties we used. Additionally not all simplices satisfying Sperner's Lemma can be applied to our work.

The connection between the constructive procedures found in our work and that of Lemke and Howson [Lemke and Howson Jr, 1964] has been well documented throughout this Thesis by comparison to the description of the Lemke-Howson procedure produced by Shapley [Shapley, 1974]. In particular our work is an exact generalisation. In Chapter 1.7 we included a description of the proofs by Roseunmüller, [Rosenmüller, 1971], and Wilson, [Wilson, 1971]. Recall these proofs provided a constructive procedure for finding Nash equilibrium situations in a N-player non-degenerate game which once again relied upon the use of completely labelled paths. We believe our work is also a direct extension of theirs too, but we have not attempted to answer this question here.

Moving away from the constructive procedures and looking at the underlying proofs is where the main differences arise. In summary our proof removes the remaining geometric considerations currently used. In particular, the proofs of Nash's Theorem published by Lemke and Howson, [Lemke and Howson Jr, 1964], Rosumüller, [Rosenmüller, 1971], Wilson, [Wilson, 1971], and Harsanyi, [Harsanyi, 1973] rely on the construction of a certain geometric object, of course with identifiable differences for each author. The approach taken by Lemke and Howson, Rosumüller and Wilson resulted in the construction of a geometric object formed by considering strategies which satisfy a specified subset of the criteria for an equilibrium situation. Edges and nodes were shown to be a subset of situations contained in the geometric object which satisfied a higher number of these conditions. Harsanyi took a different approach and achieved his geometric object by defining a class of logarithmic functions with parameter t. By looking at the combined solution set of the games for $0 \leq t \leq 1$ branches were identified with definable end points.

In all cases paths were constructed by connecting adjacent branches, or edges, which shared the same end point. Since any node is the end point of at most two edges the paths are well defined and do not allow for ambiguity with the route which should be taken. Equilibrium situations are then those end points which belong to just one arc or branch; this is true for all approaches to the construction of the geometric object.

The condition of their being a finite number of equilibrium situations was attained by a simple observation. All equilibrium situations are a subset of all nodes in the geometric object. Since the total number of nodes is finite then so must the total number of equilibrium situations. This then leaves the proof of their being an odd number of such situations in the non-degenerate case. In the papers by Lemke and Howson, Rosenmüller and Wilson unbounded edges, that is those edges with just one end point, were introduced. By showing there are an odd number of such unbounded edges and that any end point of a path is an equilibrium situation, the paths containing unbounded edges must identify an odd number of equilibrium points. All other paths with defined end points would introduce two additional equilibrium situations. Therefore if such paths exist then they will have the affect of increasing the total number of equilibrium situations by an even number and so the total number of equilibrium situations must be odd.

The proofs presented by Rosenmüller and Wilson use induction to obtain their result, with both mathematicians using the proof by Lemke and Howson to verify the result for the two player game. By assuming there are an odd number of equilibrium situations in the subgame involving $N-1$ players, Rosenmüller and Wilson were able to ensure the oddness result in the N-player game. In contrast, the proof by Harsanyi proved there was just one equilibrium situation for the class of games when $t = 1$. This was then shown to be an end point of a branch which terminated with an equilibrium point of the original game. All other branches with distinct end points identify two equilibrium situations in the game under consideration so once again identifying an odd number of equilibrium situations.

The underlying process of our proof of Nash's Theorem does follow a similar structure. We have described a geometric object and created a series of paths which have allowed us to determine there are a finite and odd number of equilibrium situations in the non-degenerate game. Our geometric object is constructed by considering optimal strategies for each player or alternatively we restrict ourselves to a sub-game for which at least one player the conditions of an equilibrium situation are met. This is much less restrictive than previous proofs.

While the foundations and interpretations of our geometric object are similar to those seen previously we do not interpret the information in the same way. Where previous proofs have defined a set of equations and inequalities which need to be satisfied we have interpreted our conditions in terms of a covering. By observing the most important properties of this covering are the points of intersection and the relationship between the covering elements (and boundary faces) we translated our covering into a well defined geometric object, a simplicial complex. This is in comparison to previous proofs where for the bimatrix case Lemke and Howson defined a convex polyhedron and for N-player games Rosenmüller and Wilson made use of some real algebraic geometry . The lack of equation set to govern our geometric object first of all allowed an insight to a more generalised model and secondly lead us to a combinatoric proof.

From the onset of construction it is clear our geometric object, η, is overly complicated and so we simplify matters by defining a graph G which contains all the paths connecting equilibrium and sub-equilibrium simplices. Again similarities are appearing and to compound this we insist the graph is of degree 2 which coincides with the requirement that nodes are the end points of at most two arcs and like before a given vertex belongs to just one unique path. However our paths have one substantial benefit. They are not governed by a set of equations or inequalities, instead the graph can be considered as it is without the need for identifying edges and nodes as those situations from P which satisfy a tighter set of inequalities. Therefore our graph is a simpler geometric representation than those seen before.

The vertices of our graph are considered in terms of being normal and extreme, where these properties are not mutually exclusive and a vertex can satisfy both conditions simultaneously. Once again we look at the degree of these nodes to establish certain properties about them. Rather than demonstrating there are an odd number of paths which contain just 1 equilibrium situation and all others contain two, we make use of the properties of a graph and Lemma 2.22. This is a combinatorial result which does not require the tracing of some finite path through a convex polyhedron nor does the proof need to be completed in stages. Previous proofs require the identification of an odd number of paths which contain just one equilibrium point and then show any other equilibrium point occurs in a pair. This procedure is all contained within Lemma 2.22 which leaves us just to determine there are an odd number of vertices satisfying the condition to be extreme and so reduces the steps in the final proof.

The finite result comes directly from the resulting simplicial complex being bounded and containing a finite number of simplices and thus there are a finite number of vertices in G. Since all equilibrium situation are a subset of these vertices we must have a finite number of equilibrium situations. Our proof also uses induction on the size of the game which clearly mimics the approach by Rosenumüller and Wilson.

The main difference our approach offers is in the construction and interpretation of the conditions defining our geometric object, but in doing this our proof also encounters a difficulty not previously encountered. In the proofs discussed in Section 1.7 the requirements that paths are unique and no node is the end point for more than two of the edges was a natural consequence of the geometric object defined. For our work we required this property of our graph G. Recall the vertices of G correspond to equilibrium or sub-equilibrium simplices from η and an edge connects two vertices if the corresponding simplices share an equilibrium face. Thus G will only be of degree 2 if and only if a sub-equilibrium face belongs to at most two simplices. This is equivalent to the property of non-ramification. However we have provided examples where this situation does not naturally hold and ultimately this will prevent the use of Lemma 2.22 to determine Nash's Theorem. We overcame this problem by observing, in the non-degenerate case, the nerves can be regularised without affecting the parity of equilibrium situations it contained. This step was not needed previously. The reasons for this complication lies in the construction of the simplicial complexes which are more general than the convex polyhedra seen in previous proofs.

The paths, defined to the be union of 1-dimensional adjacent edges, as described by Lemke-Howson, Rosenmüller and Wilson are originally defined to be points satisfying partial equilibrium situation conditions. However observe such paths can also be identified in our approach as the intersection of covering elements of a finite covering of P. Consequently the paths can be interpreted as a subcomplex of a certain simplicial complex, namely a modified nerve of the covering. This leads us to the largest and most important difference between our proof and those described in Section 1.7

We have presented a generalised model defined completely independently from any notion of game theory and it was this which was used to determine the oddness criteria in the way described above. In particular we started by defining a simplicial complex $\mathcal{S}$, as economically as we can, so that on one hand, the analogy of Nash's Theorem still holds, and, on the other, the traditional case of a non-cooperative game is covered. Within the simplicial complex setting no properties of game theory have been used to prove the more general result and so rather than prove non-degenerate games have a finite and odd number of equilibrium points we have shown this property to be true in a much broader setting. In particular the existence of equilibria for $\mathcal{S}$ allows us to extend Nash's Theorem to generalised games in which mixed strategy payoff functions are no longer required to be polylinear and may in fact be replaced by some total orderings. In such generalised games the paths defined by Lemke-Howson, Rosenmüller and Wilson may no longer exist in the naive sense (as demonstrated in Example 7.3) but are replaced by their interpretations as sub-complexes of nerves of coverings. As a result of this we have demonstrated Nash's Theorem, as traditionally given, originates from underlying combinatoric properties of a much more general model and is not a result confined to game theory.

Chapter 9

Further Work

In this Thesis we have proved Nash's Theorem can be extended and applied in a mathematical setting much larger than previously published work allows. In doing this we have achieved a result which we hope will have implications and consequences for a wide range of applications. Staying with game theory the range of non-cooperative games which are now guaranteed to contain an equilibrium situation have increased. Therefore if the situation under consideration can be modelled more accurately and realistically with payoff functions which are not polylinear, then this can be done with the assurance that the game will still satisfy Nash's Theorem. Of course such games will be abstract in nature. Once Nash's Theorem can be applied to a game then we are automatically assured we can identify at least one equilibrium situation, or solution, without risk of failure.

Having shown the existence of equilibrium situations is a result of an underlying combinatoric property exhibited by game theory, our work may additionally provide an avenue to explore other areas of current research in game theory. These may include the development of efficient algorithms to identify equilibrium situations or for greater understanding of the topology of equilibrium situations in the non-degenerate and degenerate case.

The representation of non-cooperative game theory we have used in this Thesis differs greatly to the formation traditionally used. As such it may therefore be possible to translate and interpret other branches of game theory in this way, for example coalition games where two or more players work together to achieve optimal payoffs. Our Thesis also supports previous work and demonstrate the 2-player non-cooperative game is substantially simpler than the case for N-players. Examining the properties of functions (satisfying the definition to be a total order) may identify additional sub-classes of games which are simpler and easier to analyse.

Chapter 5 on bimatrix games provides an insight to another avenue of investigation. In the game Γ the nerve η can be represented as the join $\eta_1 * \eta_2$. However when we extend this to our generalised game we can no longer guarantee η is a non-ramified complex. Of course in this case η will still be of the correct dimension. However because η failing to be a non-ramified complex must result in either η_1 or η_2 failing this condition too, it is not clear how regularisation can be used here. To rectify this problem it appears a modification to the covering needs to be made. To ensure all intersections are maintained it seems unlikely this will cause a sufficient perturbation of the covering elements. Therefore it seems the correct approach would be the introduction of tubular neighbourhoods at the points of intersection. However this still may not provide sufficient protection against the associative nerve failing the conditions of non-ramification (the result of a disjoint intersection). Dividing the covering elements (first before the introduction of neighbourhoods) and taking each segment as a new covering element may help matters, for example by constructing a triangulation of each covering element. This appears to be more complicated than the proof provided in this Thesis but perhaps should not be discounted. Clearly the form of the coverings is important, but can we find a method or process to simplify the intersections of the covering elements without losing information about the corresponding situation? Alternatively is there another way to define the nerve η such that in the non-degenerate case it will always satisfy the conditions of non-ramification without any need for regularisation?

Our work has demonstrated Nash's Theorem is not a consequence of game theory but instead is the result of underlying combinatoric properties of a much larger mathematical model. Therefore the consequences of this Thesis are unlikely to be constrained to game theory but may have implications in other branches of mathematics. By equation (1.25) and Definition 1.10 an equilibrium situation of a non-cooperative game is the solution to a finite set of inequalities and equations and as such the set of all equilibrium situations is necessarily a semi-algebraic set. It is then natural to ask if our generalisation to game theory can at the very least replicate current results within semi-algebraic geometry.

The key result in proving Nash's Theorem for simplicial complexes is an abstract version of Sperner's Lemma. This leads us to realise that our simplicial complex also contains Sperner's Lemma as a particular example. Equivalently, as in the game theory case, our simplicial complex provides a much larger model which generalises this result. In particular we can deduce the underlying mathematical properties which allow Nash's Theorem and Sperner's Lemma to be true are shared and as such they are both the result of the same mathematical phenomena. Therefore on analysing the simplicial complex model it maybe possible to identify other branches of mathematics, or possibly other unrelated theory, which prove to be an special example of our simplicial complex and thus forging an unexpected connection to both Sperner's Lemma and Nash's Theorem.

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